INFINITE SQUARE WELL - COMBINATION OF TWO LOWEST STATES

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Reference: Griffiths, David J. (2005), Introduction to Quantum Mechanics, 2nd Edition; Pearson Education - Problem 2.5.

As an example of an explicit case of a particle in the infinite square well, suppose we have a particle that starts off in a combination of the two lowest states:

$$\Psi(x,0) = A\left[\psi_1(x) + \psi_2(x)\right] \tag{1}$$

To normalize, we find A by using the orthonormal property of the stationary states, so:

$$\begin{aligned}
\int |\Psi(x,0)|^2 dx &= A^2 \int (\psi_1^* + \psi_2^*)(\psi_1 + \psi_2)dx && (2)\\
&= A^2 \int (|\psi_1|^2 + |\psi_2|^2)dx && (3)\\
&= 2A^2 && (4)\\
&= 1 && (5)
\end{aligned}$$

So $A = 1/\sqrt{2}$.

Using $\omega \equiv \pi^2\hbar/2ma^2$, we have for the full wave function:

$$\Psi(x,t) = \frac{\sqrt{2}}{2}\psi_1(x)e^{-i\omega t} + \frac{\sqrt{2}}{2}\psi_2(x)e^{-4i\omega t} \tag{6}$$

and

$$\begin{aligned}
2|\Psi(x,t)|^2 &= (\psi_1^*(x)e^{i\omega t} + \psi_2^*(x)e^{4i\omega t})(\psi_1(x)e^{-i\omega t} + \psi_2(x)e^{-4i\omega t}) && (7)\\
&= \psi_1^2 + \psi_2^2 + 2\psi_1\psi_2\cos(3\omega t) && (8)\\
|\Psi(x,t)|^2 &= \frac{1}{2}(\psi_1^2 + \psi_2^2 + 2\psi_1\psi_2\cos(3\omega t)) && (9)
\end{aligned}$$

The stationary states are real functions, so we can drop the * notation on the complex conjugates. Note that $\int|\Psi(x,t)|^2dx=1$ for all times, since the cosine term integrates to zero due to orthogonality.

The average position is:

$$\begin{aligned}\langle x\rangle &= \frac{1}{2}\int_0^a\left(x\psi_1^2(x)+x\psi_2^2(x)+2x\psi_1\psi_2\cos(3\omega t)\right)dx &\quad(10)\\ &= \frac{a}{2}-\frac{16a}{9\pi^2}\cos(3\omega t) &\quad(11)\end{aligned}$$

The particle's mean position oscillates about the midpoint of the well with an amplitude of $16a/9\pi^2\approx0.18a$.

The mean momentum can be found the quick way by taking the derivative of $\langle x\rangle$.

$$\langle p\rangle=m\frac{d\langle x\rangle}{dt}=\frac{8\hbar}{3a}\sin(3\omega t) \quad (12)$$

where we have used the definition of ω. Doing it the long way using integration does give the same answer, as can be checked using Maple (or by hand).

The two possible energies are E_1 and E_2 and since the wave function consists of equal contributions from the corresponding stationary states, they occur with equal probability. Thus

$$\langle H\rangle=\frac{1}{2}(E_1+E_2)=\frac{5\pi^2\hbar^2}{4ma^2} \quad (13)$$

Again, this can be obtained the long way through integration.

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