COMMUTATORS: A FEW THEOREMS

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Post date: 17 Sep 2012.
The commutator of two operators is defined as

\[ [A, B] \equiv AB - BA \quad (1) \]

In general, a commutator is non-zero, since the order in which we apply operators can make a difference. In practice, to work out a commutator we need to apply it to a test function \( f \), so that we really need to work out \([A, B]f\) and then remove the test function to see the result. This is because many operators, such as the momentum, involve taking the derivative.

We’ll now have a look at a few theorems involving commutators.

Theorem 1:

\[ [AB, C] = A[B, C] + [A, C] B \quad (2) \]

Proof: The LHS is:

\[ [AB, C] = ABC - CAB \quad (3) \]

The RHS is:

\[ A[B, C] + [A, C] B = ABC - ACB + ACB - CAB = ABC - CAB = [AB, C] \quad (6) \]

QED.

Theorem 2:

\[ [x^n, p] = i\hbar nx^{n-1} \quad (7) \]

where \( p \) is the momentum operator.

Proof: Using \( p = \frac{\hbar}{i} \partial / \partial x \) and letting the commutator operate on some arbitrary function \( g \):
\[ [x^n, p] g = x^n \frac{\hbar}{i} \frac{\partial}{\partial x} g - \frac{\hbar}{i} \frac{\partial}{\partial x} (x^n g) \] (8)

\[ = x^n \frac{\hbar}{i} \frac{\partial}{\partial x} - \frac{\hbar}{i} nx^{n-1} g - x^n \frac{\hbar}{i} \frac{\partial}{\partial x} \] (9)

\[ = i\hbar nx^{n-1} g \] (10)

Removing the function \( g \) gives the result \([x^n, p] = i\hbar nx^{n-1}. \) QED.

**Theorem 3:**

\[ [f(x), p] = i\hbar \frac{df}{dx} \] (11)

Again, letting the commutator operate on a function \( g \):

\[ [f(x), p] = f \frac{\hbar}{i} \frac{\partial}{\partial x} g - \frac{\hbar}{i} \frac{\partial}{\partial x} (fg) \] (12)

\[ = f \frac{\hbar}{i} \frac{\partial}{\partial x} g - \frac{\hbar}{i} \frac{\partial f}{\partial x} g - f \frac{\hbar}{i} \frac{\partial g}{\partial x} \] (13)

\[ = i\hbar \frac{\partial f}{\partial x} g \] (14)

Removing \( g \) gives the result \([f(x), p] = i\hbar \frac{\partial f}{\partial x}. \) QED.

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