CAYLEY TABLES FOR FINITE GROUPS

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Reference: *A Gentle Introduction to Group Theory*, Bana Al Subaiei & Muneerah Al Nuwairan, Section 5.2.

A *Cayley table* is a visual aid to the binary operations in a finite group, usually one whose underlying set contains a relatively small number of elements. It's useful to illustrate the actions of the binary operation in cases where that action can't be easily written as a mathematical formula. The table lists the results of applying the binary operation with the first element in the operation given in the leftmost column and the second element in the top row.

Example 1. Write the Cayley table for the group $(\mathbb{Z}_5, \oplus_5)$. The set $\mathbb{Z}_5$ is the set of equivalence classes of the integers modulo 5. The operation $\oplus_5$ is addition modulo 5. For example, $[3] \oplus_5 [4] = [2]$ since $(3+4) \mod 5 = 7 \mod 5 = 2$.

The table is shown in Table 1. From the table, we can see by inspection that the result of $a \oplus_5 b$ gives another element in the set for every a, b, so the group has closure. We can also see that the identity element is [0] since $[0] \oplus_5 b = b$ for all b and $a \oplus_5 [0] = a$ for all a. We can also see that every element has an inverse, since the element [0] appears in every row and column. Thus for example $[2] \oplus_5 [3] = [0]$. Also, since the table is symmetric about the main diagonal, this group is commutative.

a \ b	[0]	[1]	[2]	[3]	[4]
[0]	[0]	[1]	[2]	[3]	[4]
[1]	[1]	[2]	[3]	[4]	[0]
[2]	[2]	[3]	[4]	[0]	[1]
[3]	[3]	[4]	[0]	[1]	[2]
[4]	[4]	[0]	[1]	[2]	[3]

TABLE 1. Cayley table for $(\mathbb{Z}_5, \oplus_5)$ with $a \oplus_5 b$.

$a \backslash b$	[0]	[1]	[2]	[3]
[0]	[0]	[0]	[0]	[0]
[1]	[0]	[1]	[2]	[3]
[2]	[0]	[2]	[0]	[2]
[3]	[0]	[3]	[2]	[1]

TABLE 2. Cayley table for $(\mathbb{Z}_4, \otimes_4)$ with $a \otimes_4 b$.

$a \backslash b$	[1]	[3]
[1]	[1]	[3]
[3]	[3]	[1]

TABLE 3. Cayley table for subset of $(\mathbb{Z}_4, \otimes_4)$ with $a \otimes_4 b$.

Unfortunately, there's no easy way to check associativity using a Cayley table. We need to check that $(a \oplus_5 b) \oplus_5 c = a \oplus_5 (b \oplus_5 c)$ for all combinations of a, b, c. This can be done, but even in this relatively simple table, this would require checking all $5^3 = 125$ combinations, so is not a trivial task. In this case, we can infer associativity from the associativity of the modulus operation, but in the general case, we'd need to resort to another method.

Example 2. The Cayley table can also be used to show that a binary operation does not give a group. The Cayley table for $(\mathbb{Z}_4, \otimes_4)$ uses multiplication modulo 4, so we get Table 2.

The binary operation does have closure, as every operation $a \otimes_4 b$ gives another element in the set. The element $[1]$ is the identity, but not every element has an inverse, as there is no element which can be combined with either [0] or [2] to give $[1]$. Thus this is not a group.

However, we can form a subset of $(\mathbb{Z}_4, \otimes_4)$ containing only $[1]$ and $[3]$ to give Table 3

In this table, we have closure and an identity in the element $[1]$. Each of the two elements $[1]$ and $[3]$ has an inverse so (assuming we've checked associativity) this subset does form a group.

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