KLEIN-GORDON EQUATION: PLANE WAVE SOLUTIONS

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The Klein-Gordon equation was one of the first attempts at producing a relativistic quantum theory. In natural units, the equation is

\[(\partial_{\mu}\partial^{\mu} + m^2)\phi = 0\] (1)

This equation also results from the Euler-Lagrange equation for a scalar field \(\phi\) with Lagrangian

\[\mathcal{L} = \frac{1}{2} (\partial_{\mu}\phi)(\partial^{\mu}\phi) - \frac{1}{2} m^2 \phi^2\] (2)

This is the Lagrangian for zero potential \(V(\phi) = 0\).

To write solutions to equation (1), we can introduce some new notation. In natural units, the four-momentum is

\[p_{\mu} = \begin{bmatrix} E \\ p_i \end{bmatrix}\] (3)

\[= \begin{bmatrix} E \\ -p^i \end{bmatrix}\] (4)

The scalar product of four-momentum with a spacetime vector \(x^{\mu}\) is therefore

\[px \equiv p_{\mu}x^{\mu} = Et - \mathbf{p} \cdot \mathbf{x}\] (5)

For a plane wave with angular frequency \(\omega\), Planck’s relation is \(E = \hbar \omega = \omega\), and the wave vector \(\mathbf{k}\) has components in the three spatial directions of \(2\pi/\lambda_i\), where \(\lambda_i\) is the component of the wavelength in direction \(x_i\). For example, a wave moving in the \(x_1\) direction has \(\lambda_2 = \lambda_3 = \infty\), so \(\mathbf{k} = [k_1, 0, 0]\). The four-vector \(k^{\mu}\) is

\[k^{\mu} = [\omega, \mathbf{k}]\] (6)

and since \(\mathbf{p} = \mathbf{k}\) in natural units, we have

\[k \cdot x = k_{\mu}x^{\mu} = p_{\mu}x^{\mu} = px\] (7)

A plane wave solution to (1) turns out to be
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\[
\phi = \sum_k \frac{1}{\sqrt{2V\omega_k}} \left( A_k e^{-ikx} + B_k^\dagger e^{ikx} \right) 
\]  

(8)

We can see this by direct substitution. Consider one term \( \phi_k \) from the sum. Then

\[
\phi_k = \frac{1}{\sqrt{2V\omega_k}} \left( A_k e^{-ikx} + B_k^\dagger e^{ikx} \right) 
\]  

(9)

\[
\partial\mu \phi_k = \frac{1}{\sqrt{2V\omega_k}} \left( -ik\mu A_k e^{-ikx} + ik\mu B_k^\dagger e^{ikx} \right) 
\]  

(10)

\[
\partial\mu \partial\mu \phi_k = \frac{1}{\sqrt{2V\omega_k}} \left( -k\mu k\mu A_k e^{-ikx} - k\mu k\mu B_k^\dagger e^{ikx} \right) 
\]  

(11)

However, using the invariant scalar \( p_\mu p^\mu \) from relativity:

\[
k\mu k^\mu = p_\mu p^\mu = E^2 - p^2 = m^2 
\]  

(12)

Thus

\[
\partial\mu \partial\mu \phi_k = -m^2 \phi_k 
\]  

(13)

so \( \Box \) is true for a single component \( \phi_k \). Since the solution \( \phi_k \) is a linear combination of such solutions, and the original differential equation is linear, then \( \phi_k \) is also a solution. [The normalization factor \( 1/\sqrt{2V\omega_k} \) is irrelevant in proving that \( \phi_k \) is a solution; it’s just there to make future calculations easier.]

Note that the first term (involving \( A_k \)) is also a solution of the free-particle Schrödinger equation, which is

\[
i \frac{\partial \phi_S}{\partial t} = -\frac{1}{2m} \nabla^2 \phi_S 
\]  

(14)

If we take

\[
\phi_S = \sum_k A_k e^{-ikx} 
\]  

(15)

then

\[
\frac{\partial \phi_S}{\partial t} = \sum_k \nabla^2 \phi_S = \sum_k \frac{k^2}{2m} A_k e^{-ikx} 
\]  

(16)
For a free particle, the energy $E_k = \frac{p^2}{2m} = \frac{k^2}{2m}$, so the Schrödinger equation is satisfied. The second term (with $B_k^\dagger$) does not satisfy the Schrödinger equation, since in that case we get

$$\phi_S = \sum_k B_k^\dagger e^{ikx} \quad (18)$$

$$i\frac{\partial \phi_S}{\partial t} = -\sum_k E_k B_k^\dagger e^{ikx} \quad (19)$$

$$-\frac{1}{2m} \nabla^2 \phi_S = \sum_k \frac{k^2}{2m} B_k^\dagger e^{ikx} \quad (20)$$

The extra minus sign means the two sides don’t match.

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