KLEIN-GORDON EQUATION: ORTHONORMALITY OF SOLUTIONS

Link to: physicspages home page.
To leave a comment or report an error, please use the auxiliary blog.
Post date: 5 Dec 2015.

The plane wave solutions of the Klein-Gordon equation are

\[ \phi = \sum_k \frac{1}{\sqrt{2V\omega_k}} \left( A_k e^{-ikx} + B_k^\dagger e^{ikx} \right) \]  \hspace{1cm} (1)

We can redefine a couple of terms by introducing

\[ \phi_{k,A} \equiv \frac{e^{-i k x}}{\sqrt{V}} \]  \hspace{1cm} (2)
\[ \phi_{k,B}^\dagger \equiv \frac{e^{i k x}}{\sqrt{V}} \]  \hspace{1cm} (3)

Then

\[ \phi = \sum_k \frac{1}{\sqrt{2\omega_k}} \left( A_k \phi_{k,A} + B_k^\dagger \phi_{k,B}^\dagger \right) \]  \hspace{1cm} (4)

The \( \phi_{k,A} \) and \( \phi_{k,B}^\dagger \) are orthonormal functions. We have

\[ \int \phi_{k,A}^\dagger \phi_{k',A} d^3x = \frac{1}{V} \int e^{i(k'-k)x} d^3x \]  \hspace{1cm} (5)

where the integral is over the volume \( V \), and the wavelengths of the plane waves fit an integral number of times within \( V \), so that the amplitudes of the waves at the boundaries are all zero. The four-vector \( k \) is defined as

\[ k = [\omega_k, \mathbf{k}] \]  \hspace{1cm} (6)

If \( k' = k \), the integrand is 1 and is integrated over \( V \), so the result is

\[ \int \phi_{k,A}^\dagger \phi_{k,A} d^3x = \frac{1}{V} \int e^{i(k'-k)x} d^3x \]  \hspace{1cm} (7)
\[ = \frac{V}{V} = 1 \]  \hspace{1cm} (8)
If \( k' \neq k \), consider the integral over \( x^1 = x \) (for the purposes of this derivation only, \( x \) refers to the single \( x \) dimension of the 3-vector \( \mathbf{x} \) and should not be confused with the four-vector \( \mathbf{x} \) used in \( \mathbf{x} \)):

\[
\int \phi^\dagger_{k',A} \phi_{k,A} dx = \frac{1}{V} e^{i(\omega_{k'} - \omega_k) t} e^{-i(k_y - k'_y) y} e^{-i(k_z - k'_z) z} \int e^{-i(k_x - k'_x) x} dx
\]

(9)

\[
= -\frac{1}{i(k_x - k'_x) V} e^{i(\omega_{k'} - \omega_k) t} e^{-i(k_y - k'_y) y} e^{-i(k_z - k'_z) z} \left[ e^{-i(k_x - k'_x) x} \right]_{x=x_0}^{x=x_1}
\]

(10)

\[
= 0
\]

(11)

where \( x_0 \) and \( x_1 \) are the \( x \) limits of \( V \), where by assumption the wave amplitude is zero. Therefore

\[
\int \phi^\dagger_{k',A} \phi_{k,A} d^3 x = \delta_{k,k'}
\]

(12)

The same result follows for \( \phi_{k,B}^\dagger \) by just replacing \( kx \) by \(-kx\) throughout the derivation, so

\[
\int \phi^\dagger_{k,B} \phi_{k,B}^\dagger d^3 x = \delta_{k,k'}
\]

(13)

For mixed terms, we have

\[
\int \phi^\dagger_{k',A} \phi_{k,B}^\dagger d^3 x = \frac{1}{V} \int e^{i(k'x + k_x) x} d^3 x
\]

(14)

In this case, the exponent cannot be zero, so the integral always comes out to zero, so that

\[
\int \phi^\dagger_{k',A} \phi_{k,B}^\dagger d^3 x = 0
\]

(15)

PINGBACKS

Pingback: Klein-Gordon equation: probability density and current