CREATION AND ANNIHILATION OPERATORS: NORMALIZATION

The number operators are defined in terms of the creation and annihilation operators for the free scalar Hamiltonian as

\[ N_a(k) = a^\dagger(k) a(k) \quad (1) \]
\[ N_b(k) = b^\dagger(k) b(k) \quad (2) \]

We’ve seen that \( a^\dagger(k) \) creates a particle of energy \( \omega_k \) when it operates on a state, and \( a(k) \) destroys a particle with energy \( \omega_k \) when it operates on a state (if that state contains such a particle). That is, the state \( a^\dagger(k) | n_k \rangle \) is an eigenstate of \( N_a(k) \) with eigenvalue \( n_k + 1 \) and \( a(k) | n_k \rangle \) is an eigenstate of \( N_a(k) \) with eigenvalue \( n_k - 1 \). However, if we require all multiparticle states to be normalized, so that \( \langle n_k | n_k \rangle = 1 \), the states \( a^\dagger(k) | n_k \rangle \) and \( a(k) | n_k \rangle \) do not produce normalized states. Rather, we have

\[ a^\dagger(k) | n_k \rangle = A | n_k + 1 \rangle \quad (3) \]
\[ a(k) | n_k \rangle = B | n_k - 1 \rangle \quad (4) \]

for some constants \( A \) and \( B \) that are determined by requiring normalization.

To find \( A \) and \( B \), we can take the modulus of the states above. We get (we’ll leave off the \( (k) \) dependence of the \( a^\dagger \) and \( a \) operators to save typing; everything in what follows occurs at wave number \( k \); we’ll also assume \( A \) and \( B \) are real, although they could also be multiplied by some phase factor \( e^{i\alpha} \), but this just complicates things unnecessarily). By using the commutation relation

\[ [a, a^\dagger] = 1 \quad (5) \]

we get, from \(^3\)
\[ \langle n_k | aa^\dagger | n_k \rangle = A^2 \langle n_k + 1 | n_k + 1 \rangle = A^2 \] (6)
\[ \langle n_k | a^\dagger a | n_k \rangle = \langle n_k | a^\dagger a + 1 | n_k \rangle \] (7)
\[ \quad = \langle n_k | N_a + 1 | n_k \rangle \] (8)
\[ \quad = \langle n_k + 1 | n_k | n_k \rangle \] (9)
\[ \quad = (n_k + 1) \] (10)
\[ A = \sqrt{n_k + 1} \] (11)

Therefore
\[ a^\dagger (k) | n_k \rangle = \sqrt{n_k + 1} | n_k + 1 \rangle \] (12)

For the annihilation operator, we have from (4):
\[ \langle n_k | a^\dagger a | n_k \rangle = B^2 \langle n_k - 1 | n_k - 1 \rangle = B^2 \] (13)
\[ \langle n_k | a^\dagger a | n_k \rangle = \langle n_k | N_a | n_k \rangle \] (14)
\[ \quad = n_k \langle n_k | n_k \rangle \] (15)
\[ \quad = n_k \] (16)
\[ B = \sqrt{n_k} \] (17)

Therefore
\[ a (k) | n_k \rangle = \sqrt{n_k} | n_k - 1 \rangle \] (18)

This relation implies that applying \( a (k) \) to a state that contains no particles with energy \( \omega_k \) (that is, where \( n_k = 0 \)) produces 0. In particular, if we apply \( a (k) \) to the vacuum state, we end up with no state at all:
\[ a (k) | 0 \rangle = 0 \] (19)

Note that \( | 0 \rangle \) and 0 aren’t the same thing: \( | 0 \rangle \) is a quantum state with no particles in it, while 0 is mathematically zero, that is, nothing. As an analogy, \( | 0 \rangle \) is like having a bucket with nothing in it, while 0 corresponds to removing the bucket as well.

We can repeat exactly the same calculations for the antiparticle operators \( b^\dagger \) and \( b \) and get the results
\[ b^\dagger (k) | \bar{n}_k \rangle = \sqrt{\bar{n}_k + 1} | \bar{n}_k + 1 \rangle \] (20)
\[ b (k) | \bar{n}_k \rangle = \sqrt{\bar{n}_k} | \bar{n}_k - 1 \rangle \] (21)
PINGBACKS

Pingback: [Feynman propagator for scalar fields](#)