In our earlier look at the Dirac equation, we simply stated the equation and then derived a few properties of its components. It seems that this is pretty much the way Dirac himself derived it. He was seeking a relativistic generalization of the Schrödinger equation, so he wanted his equation to have the form

\[ i \frac{\partial}{\partial t} |\psi\rangle = H |\psi\rangle \]  

for some Hamiltonian \( H \). The original Schrödinger equation defined \( H \) (for a free particle) as \( H = -\frac{\nabla^2}{2m} \), so this non-relativistic equation had a first-order time derivative and a second order spatial derivative. The Klein-Gordon equation contains second-order derivatives in both time and space.

The problem Dirac faced was therefore how to define \( H \) to make relativistic and at the same time, retain the first order time derivative. In special relativity, energy, mass and momentum are related by

\[ E^2 = p^2 + m^2 \]  

If we use the operator form for the momentum: \( p^i = -i \partial^i \), then attempting to use \( E = \sqrt{p^2 + m^2} \) as the Hamiltonian operator runs into the problem that the momentum operator is inside a square root. Dirac solved this problem by proposing that

\[ H = \alpha \cdot p + \beta m \]  

where \( \alpha \) is a 3-d vector of matrices and \( \beta \) is a single (scalar) matrix. We can then require that \( H^2 \) give the energy squared as specified in 2, which in turn imposes conditions on the matrices \( \alpha \) and \( \beta \). We saw in the earlier post that this leads to the conditions that all four matrices have zero trace, have \( \pm 1 \) eigenvalues, and have even dimension, with the minimum dimension being \( 4 \times 4 \). The matrices also satisfy anticommutation relations:
\{\alpha_i, \alpha_j\} = \{\alpha_i, \beta\} = 0 \text{ if } i \neq j \tag{4}
\alpha_i^2 = \beta^2 = I \tag{5}

Dirac and Pauli found that the smallest matrices that satisfy all these conditions are

\begin{align*}
\alpha_1 &= \begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix} \\
\alpha_2 &= \begin{bmatrix}
-\imath \\
\imath \\
-\imath \\
\imath
\end{bmatrix} \\
\alpha_3 &= \begin{bmatrix}
1 & -1 \\
1 & -1 \\
1 & -1
\end{bmatrix} \\
\beta &= \begin{bmatrix}
1 & 0 \\
0 & -1 \\
1 & 0
\end{bmatrix}
\end{align*}

(6) (7) (8) (9)

where all blank matrix elements are zero. These can be written in a more condensed form by using the Pauli spin matrices from non-relativistic quantum mechanics

\begin{align*}
\sigma_1 &= \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}; \\
\sigma_2 &= \begin{bmatrix}
0 & -\imath \\
\imath & 0
\end{bmatrix}; \\
\sigma_3 &= \begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix}
\end{align*}

(10)

We get
\[
\alpha_1 = \begin{bmatrix}
0 & \sigma_1 \\
\sigma_1 & 0
\end{bmatrix}
\] (11)

\[
\alpha_2 = \begin{bmatrix}
0 & \sigma_2 \\
\sigma_2 & 0
\end{bmatrix}
\] (12)

\[
\alpha_3 = \begin{bmatrix}
0 & \sigma_3 \\
\sigma_3 & 0
\end{bmatrix}
\] (13)

\[
\beta = \begin{bmatrix}
I & 0 \\
0 & -I
\end{bmatrix}
\] (14)

Here, each entry in each matrix is a \(2 \times 2\) submatrix given by [10].

It’s fairly obvious from their definitions that all 4 matrices have zero trace and are hermitian (that is, they equal their complex conjugate transpose). The eigenvalues of \(\beta\) are \(\pm 1\) since the matrix is diagonal. The eigenvalues of the \(\alpha_i\) can be found in the usual way. For \(\alpha_1\) we get

\[
\det(\alpha_1 - \lambda I) = \begin{vmatrix}
-\lambda & 0 & 0 & 1 \\
0 & -\lambda & 1 & 0 \\
0 & 1 & -\lambda & 0 \\
1 & 0 & 0 & -\lambda
\end{vmatrix}
\] (15)

\[
= -\lambda \left[ -\lambda \left( \lambda^2 - (-\lambda) \right) - (\lambda^2 - 1) \right] (16)
\]

\[
= \lambda^4 - 2\lambda^2 + 1 (17)
\]

\[
= (\lambda^2 - 1)^2 (18)
\]

The roots of this equation are \(\pm 1\) (twice each). The eigenvalues of \(\alpha_2\) and \(\alpha_3\) also turn out to be \(\pm 1\) as can be verified if you grind through the calculations. The conditions 4 and 5 can be verified by direct multiplication (although this is kind of tedious).