DIRAC EQUATION: 4 SOLUTION VECTORS

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Reference: Reference: Robert D. Klauber, Student Friendly Quantum Field Theory, (Sandtrove Press, 2013) - Chapter 4, Problem 4.5.
Post date: 24 Jan 2016.
The Dirac equation in relativistic quantum mechanics can be written as

\[(i \gamma^\mu \partial_\mu - m I) |\psi\rangle = 0 \quad (1)\]

When written out in its matrix components, this equation is actually 4 differential equations.

\[(i \partial_0 - m) |\psi\rangle_1 + i \partial_3 |\psi\rangle_3 + (i \partial_1 + \partial_2) |\psi\rangle_4 = 0 \quad (2)\]
\[(i \partial_0 - m) |\psi\rangle_2 + (i \partial_1 - \partial_2) |\psi\rangle_3 - i \partial_3 |\psi\rangle_4 = 0 \quad (3)\]
\[ -i \partial_3 |\psi\rangle_1 - (i \partial_1 + \partial_2) |\psi\rangle_2 - (i \partial_0 + m) |\psi\rangle_3 = 0 \quad (4)\]
\[ -i (\partial_1 + i \partial_2) |\psi\rangle_1 + i \partial_3 |\psi\rangle_2 - (i \partial_0 + m) |\psi\rangle_4 = 0 \quad (5)\]

Remember that \( |\psi\rangle \) is a 4-d column vector in spinor space rather than a single function, so that the subscript index \( j \) in \( |\psi\rangle_j \) indicates which component in spinor space we’re dealing with. These equations have four solutions denoted by \( |\psi^{(n)}\rangle \) for \( n = 1,2,3,4 \). Note that each \( |\psi^{(n)}\rangle \) is a full 4-component vector in spinor space; that is, the superscript \( (n) \) indicates which complete vector we’re dealing with. Thus \( |\psi^{(n)}\rangle_j \) is the \( j \)th component of the \( n \)th vector.

We can write the 4 PDEs as a matrix eigenvalue equation by moving the terms involving \( m \) to the RHS and factoring out an \( i \) from the terms remaining on the LHS:

\[ i \begin{bmatrix} \partial_0 & 0 & \partial_3 & \partial_1 - i \partial_2 \\ 0 & \partial_0 & \partial_1 + i \partial_2 & -\partial_3 \\ -\partial_3 & -\partial_1 + i \partial_2 & -\partial_0 & 0 \\ -\partial_1 - i \partial_2 & \partial_3 & 0 & -\partial_0 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{bmatrix} = m \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{bmatrix} \quad (6)\]

We’ll now look at the four solutions \( |\psi^{(n)}\rangle \) and verify that they satisfy \( (6) \).

First, we have
\[ |\psi^{(1)}\rangle = \sqrt{\frac{E + m}{2m}} \begin{bmatrix} 1 \\ 0 \\ \frac{p^3}{E + m} \\ \frac{p^1 + ip^2}{E + m} \end{bmatrix} e^{-ipx} \equiv u_1 e^{-ipx} \quad (7) \]

where \( u_1 \) is defined by this equation as the constant \( \sqrt{\frac{E + m}{2m}} \) multiplied by the 4-d spinor factor. Remember that \( px \) is a 4-vector product:

\[ px = p^\mu x_\mu = Et - \mathbf{p} \cdot \mathbf{x} \quad (8) \]

The derivatives in (6) are all with respect to spacetime variables, so act only on \( e^{-ipx} \); the spinor components are constants with respect to these derivatives. The first row in (6) is therefore

\[ i\sqrt{\frac{E + m}{2m}} e^{-ipx} \left[ -iE + 0 + \frac{p^3}{E + m} (ip^3) + \frac{p^1 + ip^2}{E + m} (ip^1 + p^2) \right] = \sqrt{\frac{E + m}{2m}} e^{-ipx} m \psi_1 \quad (13) \]

Thus the first row of (6) is verified. The other 3 rows can be verified similarly. For row 2:

\[ i\sqrt{\frac{E + m}{2m}} e^{-ipx} \left[ 0 + 0 + \frac{p^3}{E + m} (ip^1 - p^2) + \frac{p^1 + ip^2}{E + m} (-ip^3) \right] = 0 = m\psi_2 \quad (14) \]

For row 3:
i\sqrt{\frac{E+m}{2m}} e^{-ipx} \left[ -ip^3 + 0 + \frac{p^3}{E+m} (iE) + 0 \right] = i\sqrt{\frac{E+m}{2m}} e^{-ipx} \left[ \frac{-ip^3 (E+m) + ip^2 E}{E+m} \right]

= \sqrt{\frac{E+m}{2m}} e^{-ipx} \left[ \frac{p^2 (E+m) - (p^3 + ip E)}{E+m} \right]

= m\psi_3 \tag{17}

And for row 4:

i\sqrt{\frac{E+m}{2m}} e^{-ipx} \left[ (ip^1 + p^2) + 0 + 0 + \frac{p^1 + ip^2}{E+m} iE \right] = \left( E + m \right)

\sqrt{\frac{E+m}{2m}} e^{-ipx} \left[ \frac{(p^1 + ip^2)(E+m) - (p^3 + ip E)}{E+m} \right]

= \sqrt{\frac{E+m}{2m}} e^{-ipx} \left[ \frac{p^1 + ip^2}{E+m} \right] m = m\psi_4 \tag{21}

The other 3 solutions are

\left| \psi^{(2)} \right\rangle = \sqrt{\frac{E+m}{2m}} \left[ \begin{array}{c} 0 \\ 1 \\ \frac{p^1 - ip^2}{E+m} \\ \frac{-p^3}{E+m} \end{array} \right] e^{-ipx} \equiv u_2 e^{-ipx} \tag{22}

\left| \psi^{(3)} \right\rangle = \sqrt{\frac{E+m}{2m}} \left[ \begin{array}{c} \frac{p^3}{E+m} \\ \frac{p^1 + ip^2}{E+m} \\ 0 \\ 1 \end{array} \right] e^{ipx} \equiv v_2 e^{ipx} \tag{23}

\left| \psi^{(4)} \right\rangle = \sqrt{\frac{E+m}{2m}} \left[ \begin{array}{c} \frac{p^1 - ip^2}{E+m} \\ \frac{-p^3}{E+m} \\ 0 \\ 1 \end{array} \right] e^{ipx} \equiv v_1 e^{ipx} \tag{24}

If you really want to, you can verify that these 3 vectors satisfy 6 by grinding through the calculations as above. One point worth noting is that
the constant $\sqrt{\frac{E+m}{2m}}$ that multiplies all the solutions could be any other constant and still satisfy 6 (since the constant just cancels off both sides). It’s chosen to be $\sqrt{\frac{E+m}{2m}}$ to make later calculations easier.

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