HAMILTONIAN FOR COMPLEX SCALAR FIELD

Here we revisit the complex scalar field we considered earlier. The Lagrangian density is

\[ \mathcal{L} = \left( \partial_\mu \phi \right)^\dagger \left( \partial^\mu \phi \right) - m^2 \phi^\dagger \phi - V \left( \phi^\dagger \phi \right) \]  

(1)

where \( \phi \) is the complex field, and the potential \( V \) is a function of \( \phi^\dagger \phi \), so it is a real function. The Hamiltonian density is then defined as

\[ \mathcal{H} \equiv \Pi_A \Phi^A - \mathcal{L} \]  

(2)

where \( \Pi_A \) is the conjugate momentum for field \( \Phi^A \), defined as

\[ \Pi_A \equiv \frac{\delta \mathcal{L}}{\delta \dot{\Phi}^A} \]  

(3)

In our case, there are two fields, \( \phi \) and \( \phi^\dagger \). We can rewrite (1) as

\[ \mathcal{L} = \dot{\phi}^\dagger \dot{\phi} - |\nabla \phi|^2 - m^2 \phi^\dagger \phi - V \left( \phi^\dagger \phi \right) \]  

(4)

so the conjugate momenta are

\[ \Pi = \int d^3 y \dot{\phi}^\dagger (\mathbf{y}, t) \delta (\mathbf{x} - \mathbf{y}) \]  

(5)

\[ = \dot{\phi}^\dagger (\mathbf{x}, t) \]  

(6)

\[ \Pi^\dagger = \dot{\phi} (\mathbf{x}, t) \]  

(7)

The Hamiltonian density is therefore

\[ \mathcal{H} = \dot{\phi}^\dagger \dot{\phi} + \dot{\phi}^\dagger \dot{\phi} - \left( \dot{\phi}^\dagger \dot{\phi} - |\nabla \phi|^2 - m^2 \phi^\dagger \phi - V \left( \phi^\dagger \phi \right) \right) \]  

(8)

\[ = \dot{\phi}^\dagger \dot{\phi} + |\nabla \phi|^2 + m^2 \phi^\dagger \phi + V \left( \phi^\dagger \phi \right) \]  

(9)

\[ = |\dot{\phi}|^2 + |\nabla \phi|^2 + m^2 \phi^\dagger \phi + V \left( \phi^\dagger \phi \right) \]  

(10)