ONE-DIMENSIONAL FIELD (DISPLACEMENT OF A STRING)

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Here we’ll look at the field theory for the displacement of a string of length $l$ with fixed ends (such as a violin string). The Lagrangian (total, not just the density) is

$$L = \int_0^l dx \left[ \left( \frac{\partial u}{\partial t} \right)^2 - c^2 \left( \frac{\partial u}{\partial x} \right)^2 \right]$$  

(1)

If this string is plucked, then it will vibrate so that the displacement $u$ varies with position $x$ and time $t$. We can write the displacement as a Fourier series:

$$u(x,t) = \sum_{k=1}^{\infty} q_k(t) \sin \left( \frac{\omega_k x}{c} \right)$$  

(2)

where

$$\omega_k = \frac{\pi kc}{l}$$  

(3)

Here $k$ is an integer and $q_k$ can be thought of as a parameter which gives the contribution to the overall displacement of the pure sine wave with mode $\omega_k$. The $q_k$ are the generalized coordinates in the problem, so we can calculate the conjugate momentum using the standard formula

$$p_k = \frac{\partial L}{\partial \dot{q}_k}$$  

(4)

In our case, the Lagrangian is an integral over the square of an infinite series, so the situation might look hopeless. However, it’s not as bad as it looks, due to the fact that sine terms in the Fourier series are orthogonal functions when integrated over the interval $[0,l]$. That is
\[ 
\int_0^l \sin \left( \frac{\omega_a x}{c} \right) \sin \left( \frac{\omega_b x}{c} \right) \, dx = \int_0^l \sin \left( \frac{\pi a x}{l} \right) \sin \left( \frac{\pi b x}{l} \right) \, dx = \frac{l}{2} \delta_{ab} 
\] (5)

From [2]

\[ \frac{\partial u}{\partial t} = \sum_{k=1}^{\infty} \dot{q}_k(t) \sin \left( \frac{\omega_k x}{c} \right) \] (6)

We can ignore the second term in the integrand of 1 when calculating \( p_k \) since it doesn’t contain \( \dot{q}_k \). We therefore have

\[ p_k = \frac{\partial L}{\partial \dot{q}_k} \] (7)

\[ = 2 \int_0^l dx \frac{\partial u}{\partial t} \sin \left( \frac{\omega_k x}{c} \right) \] (8)

\[ = 2 \int_0^l dx \left( \sum_{n=1}^{\infty} \dot{q}_n(t) \sin \left( \frac{\omega_n x}{c} \right) \right) \sin \left( \frac{\omega_k x}{c} \right) \] (9)

Using [6], we see that only one of the terms (when \( n = k \)) in the sum will contribute to the integral, so we get

\[ p_k = 2 \frac{l}{2} \dot{q}_k = l \dot{q}_k \] (10)

The full Lagrangian in 1 can be worked out in a similar way. To integrate the square of 7 we use [6], so the only terms that contribute to the integral are the terms involving \( \dot{q}_k^2 \):

\[ \int_0^l dx \left( \frac{\partial u}{\partial t} \right)^2 = \sum_{k=1}^{\infty} \int_0^l \dot{q}_k^2 \sin^2 \left( \frac{\omega_k x}{c} \right) \, dx = \frac{l}{2} \sum_{k=1}^{\infty} \dot{q}_k^2 \] (11)

\[ = \frac{1}{2l} \sum_{k=1}^{\infty} p_k^2 \] (12)

The cosine is also orthogonal when integrated over \([0, l] \) so we have
\[ \int_0^l \cos \left( \frac{\omega_a x}{c} \right) \cos \left( \frac{\omega_b x}{c} \right) \, dx = \int_0^l \cos \left( \frac{\pi a x}{l} \right) \cos \left( \frac{\pi b x}{l} \right) \, dx = \frac{l}{2} \delta_{ab} \]  

We therefore get

\[ \int_0^l \, dx \, c^2 \left( \frac{\partial u}{\partial x} \right)^2 = \sum_{k=1}^{\infty} \frac{c^2 \omega_k^2}{c^2} \int_0^l q_k^2 \cos^2 \frac{\omega_k x}{c} \, dx = \frac{l}{2} \sum_{k=1}^{\infty} \omega_k q_k^2 \]  

Putting the terms together we get

\[ L = \frac{1}{2l} \sum_{k=1}^{\infty} p_k^2 - \frac{l}{2} \sum_{k=1}^{\infty} \omega_k q_k^2 \]  

The total Hamiltonian is given by

\[ H = \sum_{k=1}^{\infty} p_k q_k - L \]

\[ = \sum_{k=1}^{\infty} \frac{p_k^2}{l} - \frac{1}{2l} \sum_{k=1}^{\infty} p_k^2 + \frac{l}{2} \sum_{k=1}^{\infty} \omega_k q_k^2 \]

\[ = \frac{1}{2l} \sum_{k=1}^{\infty} \left( p_k^2 + l^2 \omega_k q_k^2 \right) \]  

We now introduce the operators (essentially the same as the raising and lowering operators for the harmonic oscillator):

\[ q_k = \sqrt{\frac{\hbar}{2l \omega_k}} \left( a^\dagger e^{i \omega_k t} + a_k e^{-i \omega_k t} \right) \]

\[ p_k = l \dot{q}_k \]

\[ = i \sqrt{\frac{l \hbar \omega_k}{2}} \left( a^\dagger e^{i \omega_k t} - a_k e^{-i \omega_k t} \right) \]  

If we now interpret \( q_k \) and \( p_k \) as operators and require the usual commutation relation between position and momentum:

\[ [q_k, p_j] = i \hbar \delta_{kj} \]
we can work out the commutators of $a_k$ and $a_k^\dagger$. We could do this by inverting the above equations to express $a_k$ and $a_k^\dagger$ in terms of $q_k$ and $p_k$, but we can also just calculate $[q_k, p_j]$ in terms of the $a_k$ and $a_k^\dagger$ operators. We have

$$[q_k, p_j] = \sqrt{\omega_j} \frac{i\hbar}{2} \left\{ [a_k, a_j^\dagger] e^{i(\omega_j - \omega_k)t} - [a_k, a_j] e^{-i(\omega_j - \omega_k)t} + [a_k^\dagger, a_j] e^{-i(\omega_j + \omega_k)t} - [a_k^\dagger, a_j^\dagger] e^{i(\omega_j + \omega_k)t} \right\}$$  

(27)

If we require this to be $i\bar{\hbar}\delta_{kj}$, then because $\omega_k$ is always positive, the middle two terms must be zero, so we have

$$[a_k, a_j] = [a_k^\dagger, a_j^\dagger] = 0$$  

(29)

Also, if $j \neq k$, then the first and last terms will have an exponential term in them, so these terms too must be zero in this case. Finally, if $j = k$, the exponentials in the first and last terms are 1, so we get in this case

$$[q_k, p_k] = \frac{i\hbar}{2} \left( [a_k, a_k^\dagger] - [a_k^\dagger, a_k] \right) = i\hbar$$  

(30)

which is true if

$$[a_k, a_k^\dagger] = 1$$  

(31)

Thus we get the final result

$$[a_k, a_j^\dagger] = \delta_{kj}$$  

(32)

Finally, we can express the Hamiltonian in terms of $a_k$ and $a_k^\dagger$ by just substituting (25) into (22). We get

$$H = \frac{1}{2\bar{\hbar}} \sum_{k=1}^{\infty} \left( -\frac{\hbar \omega_k}{2} \left[ a_k^{\dagger 2} e^{-2i\omega_k t} + a_k^{2} e^{-2i\omega_k t} - a_k^{\dagger} a_k - a_k a_k^{\dagger} \right] + \frac{\hbar \omega_k}{2} \left[ a_k^{\dagger 2} e^{2i\omega_k t} + a_k^{2} e^{2i\omega_k t} + a_k^{\dagger} a_k + a_k a_k^{\dagger} \right] + \hbar \omega_k \left( a_k^{\dagger} a_k + a_k a_k^{\dagger} \right) \right)$$  

(33)

(34)

(35)