STRESS-ENERGY TENSOR: 3 EXAMPLES

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In the derivation of Noether’s theorem, L&P define the stress-energy tensor as

\[ T^{\mu\nu} = \frac{\partial L}{\partial \left( \partial_\mu \Phi^A \right)} \partial_\nu \Phi^A - g^{\mu\nu} L \]  

(1)

where there is a sum over the index \( A \), which labels the independent fields \( \Phi^A \). \( L \) is the Lagrangian density and \( g^{\mu\nu} \) is the metric tensor in flat spacetime. The stress-energy tensor is used to define a current

\[ J^\mu = \frac{\partial L}{\partial (\partial_\mu \Phi^A)} \delta \Phi^A - T^{\mu\nu} \delta x_\nu \]  

(2)

If the Lagrangian is invariant under some transformation (that is, it has a symmetry under this transformation), then Noether’s theorem states that the corresponding current is conserved, which means that

\[ \partial_\mu J^\mu = 0 \]  

(3)

Here, we’ll work out \( T^{\mu\nu} \) for three particular Lagrangians.

Example 1. First, we’ll look at the real scalar field with the Lagrangian

\[ L = \frac{1}{2} \left( \partial_\alpha \phi \right) \left( \partial^\alpha \phi \right) - \frac{1}{2} m^2 \phi^2 - V (\phi) \]  

(4)

\[ = \frac{1}{2} \left( \partial_\alpha \phi \right) g^{\alpha\beta} \left( \partial_\beta \phi \right) - \frac{1}{2} m^2 \phi^2 - V (\phi) \]  

(5)

We can drop the index \( A \) in the first term in (4) since there is only one field. Thus we have
The tensor is then
\[
T_{\mu\nu} \equiv \partial^\mu \phi \partial^\nu \phi - g_{\mu\nu} L
\]  

Example 2. Now, we look at the complex scalar field with Lagrangian
\[
L = (\partial_\alpha \phi)^\dagger (\partial^\alpha \phi) - m^2 \phi^\dagger \phi - V(\phi^\dagger \phi) \]
\[
= (\partial_\alpha \phi)^\dagger g^{\alpha\beta} (\partial_\beta \phi) - m^2 \phi^\dagger \phi - V(\phi^\dagger \phi)
\]

We now have two fields (\phi and \phi^\dagger) so the sum over \(A\) in \[1\] contains two terms. We have
\[
\frac{\partial L}{\partial (\partial_\mu \Phi^A)} \partial^{\nu} \Phi^A = (\partial_\alpha \phi)^\dagger g^{\alpha\beta} \delta_{\beta\mu} \partial^\nu \phi + \delta_{\alpha\mu} g^{\alpha\beta} \partial_\beta \phi (\partial^\nu \phi)^\dagger
\]
\[
= (\partial^\mu \phi)^\dagger \partial^\nu \phi + \partial^\mu \phi (\partial^\nu \phi)^\dagger
\]

which gives the tensor:
\[
T^{\mu\nu} = (\partial^\mu \phi)^\dagger \partial^\nu \phi + \partial^\mu \phi (\partial^\nu \phi)^\dagger - g^{\mu\nu} L
\]

Example 3. Finally, we look at the electromagnetic Lagrangian with zero current (\(j^\mu = 0\)):
\[
L = -\frac{1}{4} F^{\alpha\beta} F_{\alpha\beta}
\]
with
\[
F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha
\]

As always, it’s important to keep track of the positions of the various indexes. Since the derivative in \[1\] involves a derivative of a field with an upper index with respect to a coordinate with lower index, we convert the Lagrangian into this form.
\[
L = -\frac{1}{4} \left( g^{\alpha\gamma} \partial_\gamma A^\beta - g^{\beta\delta} \partial_\delta A^\alpha \right) \left( g_{\beta\epsilon} \partial_\epsilon A^\gamma - g_{\alpha\phi} \partial_\phi A^\theta \right)
\]
Taking the derivative of $\mathcal{L}$ requires the product rule. We have
\[
\frac{\partial \mathcal{L}}{\partial (\partial_\mu \Phi^\rho)} = -\frac{1}{4} \left( g^{\alpha\gamma} \delta_\gamma^\mu \delta_\beta^\rho - g^{\beta\delta} \delta_\delta^\mu \delta_\alpha^\rho \right) \left( g_{\beta\epsilon} \partial_\alpha A^\epsilon - g_{\alpha\theta} \partial_{\beta} A^\theta \right) \tag{18}
\]
\[
- \frac{1}{4} \left( g^{\alpha\gamma} \partial_\gamma A^\beta - g^{\beta\delta} \partial_\delta A^\alpha \right) \left( g_{\beta\epsilon} \delta_{\alpha\mu} \delta_{\epsilon\rho} - g_{\alpha\theta} \delta_\beta^\mu \delta_\theta^\rho \right) \tag{19}
\]
Using the Kronecker deltas to reduce the indexes (including the relation $g^{\beta\delta} g_{\beta\epsilon} = \delta_\delta^\epsilon$; I’ve used two lower indexes for Kronecker deltas since for these it doesn’t matter where you place the indexes - the results are always the same) gives us
\[
\frac{\partial \mathcal{L}}{\partial (\partial_\mu \Phi^\rho)} = -\frac{1}{4} \left( \partial_\mu A_\rho - \partial_\rho A_\mu - \partial_\rho A_\mu + \partial_\rho A_\mu \right) \tag{20}
\]
\[
- \frac{1}{4} \left( \partial_\mu A_\rho - \partial_\rho A_\mu - \partial_\rho A_\mu + \partial_\mu A_\rho \right) \tag{21}
\]
\[
= - (\partial_\mu A_\rho - \partial_\mu A_\rho) \tag{22}
\]
The first term in (18) is then
\[
\frac{\partial \mathcal{L}}{\partial (\partial_\mu \Phi^\rho)} \partial_\nu A^\rho = - (\partial_\mu A_\rho - \partial_\rho A_\mu) \partial_\nu A^\rho \tag{23}
\]
\[
= - (\partial_\mu A_\rho - \partial_\rho A_\mu) \partial_\nu A_\rho \tag{24}
\]
\[
= - \mathcal{F}^{\mu\rho} \partial_\nu A_\rho \tag{25}
\]
This gives the stress-energy tensor as
\[
T^{\mu\nu} = - \mathcal{F}^{\mu\rho} \partial_\nu A_\rho - g^{\mu\nu} \mathcal{L} \tag{26}
\]