NOETHER’S THEOREM - INTERNAL SYMMETRY AND SCALED SPACETIME

As another example of an [internal symmetry], but this time combined with a change in the coordinates, we’ll consider the real scalar field with \( m = 0 \) and a potential given by \( V = \lambda \phi^4 \). The Lagrangian is

\[
\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \lambda \phi^4
\]  

(1)

If we now impose the transformations

\[
x^\mu \rightarrow bx^\mu \\
\phi \rightarrow \frac{\phi}{b}
\]

(2)

(3)

where \( b \) is a constant, then the Lagrangian becomes

\[
\mathcal{L} \rightarrow \frac{1}{b^4} \left( \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \lambda \phi^4 \right)
\]  

(4)

The first term follows from

\[
\partial_\mu = \frac{\partial}{\partial x^\mu} \rightarrow \frac{1}{b} \frac{\partial}{\partial x^\mu}
\]  

(5)

Although the Lagrangian isn’t invariant under this transformation, the action is, as it is defined as

\[
\mathcal{A} = \int d^4 x \, \mathcal{L}
\]  

(6)

and the integration increment transforms as

\[
d^4 x \rightarrow b^4 d^4 x
\]  

(7)

so that the product \( d^4 x \, \mathcal{L} \) remains invariant.

To apply [Noether’s theorem] to this symmetry, we note first that the formulas given in the textbook apply to an infinitesimal transformation, while
the constant \( b \) above is not an infinitesimal. However, we can consider the transformation where \( b = 1 - \varepsilon \), with \( \varepsilon \) an infinitesimal quantity. In this case, the transformation is infinitesimal so we can apply Noether’s theorem. In that case, to first order in \( \varepsilon \), we have

\[
x^\mu \to x^\mu - \varepsilon x^\mu \\
\phi \to \phi + \varepsilon \phi
\]

so we have

\[
\delta x^\mu = -\varepsilon x^\mu \\
\delta \phi = \varepsilon \phi
\]

The transformation in this case is an internal symmetry (since the variation occurs at a single spacetime point) but now \( \delta x^\mu \neq 0 \). We can modify L&P’s equation 2.64 by dividing the original definition of the conserved current, which is

\[
J^\mu \equiv \frac{\partial \mathcal{L}}{\partial (\partial_\mu \Phi^A)} \delta \Phi^A - T^\mu_\nu \delta x^\nu
\]

The recipe for dealing with an internal symmetry is to divide this current by the infinitesimal parameter, which in this case is \( \varepsilon \), so we get

\[
j^\mu = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \frac{\delta \phi}{\varepsilon} - \frac{T^\mu_\nu \delta x^\nu}{\varepsilon} \\
= \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \phi + T^\mu_\nu \delta x^\nu
\]

The stress-energy tensor is given by

\[
T^\mu_\nu \equiv \frac{\partial \mathcal{L}}{\partial (\partial_\mu \Phi^A)} \partial^\nu \Phi^A - g^\mu_\nu \mathcal{L}
\]

which becomes, in this case

\[
T^\mu_\nu \equiv \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \partial^\nu \phi - g^\mu_\nu \mathcal{L} \\
= \partial^\mu \phi \partial^\nu \phi - g^\mu_\nu \mathcal{L}
\]

Inserting this into (14) we have
\[ j^\mu = \phi \partial^\mu \phi + x_\nu \partial^\mu \phi \partial^\nu \phi - x_\nu g^{\mu\nu} \mathcal{L} \]  
\[ = \phi \partial^\mu \phi + x_\nu \partial^\mu \phi \partial^\nu \phi - x^\mu \mathcal{L} \]  
\[ = \partial^\nu (x_\nu \phi) \partial^\mu \phi - x^\mu \mathcal{L} \]  

where the last line condenses the second line by using the product rule.