NOETHER’S THEOREM - TRACELESS STRESS-ENERGY TENSOR

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Here’s another example of the application of Noether’s theorem. This time we consider a transformation in which the action (but not necessarily the Lagrangian) is invariant under spacetime translation and spacetime dilation. That is, the translation is given by

\[ x^\mu \rightarrow x^\mu + a^\mu \]  

(1)

where the \( a^\mu \) are constants, and the dilation is given by

\[ x^\mu \rightarrow bx^\mu \]  

(2)

where \( b \) is another constant. The field in this case is assumed not to change, that is

\[ \Phi \rightarrow \Phi \]  

(3)

For infinitesimal transformations, L&P show in their equation 2.43 that the variation in the action is given by

\[ \delta A = \int_{\Omega} d^4x \partial_\mu \left[ \frac{\partial L}{\partial (\partial_\mu \Phi^A)} \delta \Phi^A - T^{\mu\nu} \delta x_\nu \right] \]  

(4)

where the index \( A \) labels the field (we have only one field here) and \( T^{\mu\nu} \) is the stress-energy tensor. In this problem, we won’t need the explicit form of \( T^{\mu\nu} \). The integral is performed over the spacetime volume \( \Omega \) which should contain the entire region pertaining to the problem.

With the conditions above, we have \( \delta \Phi = 0 \), so the first term in the integrand of (4) is zero. To get a condition on the second term, we need \( \delta x_\nu \), which arises from the two transformations above. Since we must deal with infinitesimal translations, we assume that \( a^\mu \) is infinitesimal, and for \( b \) we have

\[ b = 1 - \varepsilon \]  

(5)
where $\epsilon$ is infinitesimal. In that case, the total infinitesimal increment is
\[ \delta x_{\nu} = a_{\nu} - \epsilon x_{\nu} \] (6)

The variation in the action is then
\[ \delta \mathcal{A} = - \int_\Omega d^4 x \, \partial_{\mu} (T^{\mu \nu} \delta x_{\nu}) \] (7)

\[ = - \int_\Omega d^4 x \, \partial_{\mu} (T^{\mu \nu} (a_{\nu} - \epsilon x_{\nu})) \] (8)

\[ = - \int_\Omega d^4 x \, \partial_{\mu} (T^{\mu \nu} (a_{\nu} - \epsilon g_{\nu \rho} x^\rho)) \] (9)

Since $a_{\nu}$ and $\epsilon$ are constants, we can evaluate the derivative to get
\[ \delta \mathcal{A} = - \int_\Omega d^4 x \left[ \partial_{\mu} T^{\mu \nu} (a_{\nu} - \epsilon g_{\nu \rho} x^\rho) - \epsilon g_{\nu \rho} T^{\mu \nu} \delta_{\mu} \right] \] (10)

\[ = - \int_\Omega d^4 x \left[ \partial_{\mu} T^{\mu \nu} (a_{\nu} - \epsilon g_{\nu \rho} x^\rho) - \epsilon g_{\nu \mu} T^{\mu \nu} \right] \] (11)

\[ = - \int_\Omega d^4 x \left[ \partial_{\mu} T^{\mu \nu} (a_{\nu} - \epsilon g_{\nu \rho} x^\rho) - \epsilon T^{\mu \mu} \right] \] (12)

We now require the action to be invariant, so that $\delta \mathcal{A} = 0$. Since $a_{\nu}$ is an arbitrary infinitesimal displacement, we must have
\[ \partial_{\mu} T^{\mu \nu} = 0 \] (13)

to eliminate the first term in (12). The other parameter $\epsilon$ is also arbitrary so anything multiplying it must also be zero. The second term in (12) thus gives us the condition
\[ T^{\mu \mu} = 0 \] (14)

That is, the stress-energy tensor is traceless in this case. Notice that this result doesn’t depend on the form of the Lagrangian or the stress-energy tensor; it is derived purely from the requirement that the action be invariant under the given transformations.