

FOCK SPACE - NUMBER OPERATOR

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Post date: 31 May 2018.

References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 3, Problem 3.6.

For the real scalar field, the vacuum state $|0\rangle$ is defined as the normalized state containing zero particles which gives zero when operated on by any annihilation operator, that is

$$a(p)|0\rangle = 0 \quad (1)$$

for any momentum p . To create a particle, we apply the creation operator, so that

$$|p\rangle = a^\dagger(p)|0\rangle \quad (2)$$

is a state containing a single particle with momentum p . Using the commutation relation

$$[a(p), a^\dagger(p')] = \delta^3(\mathbf{p} - \mathbf{p}') \quad (3)$$

we have

$$\langle p'|p\rangle = \langle 0|a(p')a^\dagger(p)|0\rangle \quad (4)$$

$$= \langle 0|\delta^3(\mathbf{p} - \mathbf{p}')|0\rangle + \langle 0|a^\dagger(p)a(p')|0\rangle \quad (5)$$

$$= \delta^3(\mathbf{p} - \mathbf{p}') + 0 \quad (6)$$

$$= \delta^3(\mathbf{p} - \mathbf{p}') \quad (7)$$

The normalized state with several particles of various momenta is given by

$$|n_1, n_2, \dots, n_i, \dots\rangle = \frac{1}{\sqrt{n_1!n_2!\dots n_i!\dots}} (\hat{a}_1^\dagger)^{n_1} (\hat{a}_2^\dagger)^{n_2} \dots (\hat{a}_i^\dagger)^{n_i} \dots |0\rangle \quad (8)$$

where $|n_1, n_2, \dots, n_i, \dots\rangle$ is the state with n_i particles with momentum p_i . The set of all such states (including the vacuum) is called *Fock space*, named after Vladimir Fock (1898 - 1974), a Soviet physicist.

In Fock space, we can define a number operator as

$$\mathcal{N} = \int d^3p a^\dagger(p) a(p) \quad (9)$$

To see that this operator counts the number of particles in a multiparticle vector, consider the state with a single particle in each of N different momentum states, that is $|p_1, p_2, \dots, p_N\rangle$. We have

$$|p_1, p_2, \dots, p_N\rangle = a^\dagger(p_1) a^\dagger(p_2) \dots a^\dagger(p_N) |0\rangle \quad (10)$$

We can now apply the number operator and see the results using 3.

$$\mathcal{N} |p_1, p_2, \dots, p_N\rangle = \int d^3p a^\dagger(p) a(p) a^\dagger(p_1) a^\dagger(p_2) \dots a^\dagger(p_N) |0\rangle \quad (11)$$

$$= \int d^3p a^\dagger(p) \left[\delta^3(\mathbf{p} - \mathbf{p}_1) + a^\dagger(p_1) a(p) \right] a^\dagger(p_2) \dots a^\dagger(p_N) |0\rangle \quad (12)$$

$$= a^\dagger(p_1) a^\dagger(p_2) \dots a^\dagger(p_N) |0\rangle + \quad (13)$$

$$\int d^3p a^\dagger(p) a^\dagger(p_1) a(p) a^\dagger(p_2) \dots a^\dagger(p_N) |0\rangle \quad (14)$$

$$= |p_1, p_2, \dots, p_N\rangle + \int d^3p a^\dagger(p) a^\dagger(p_1) a(p) a^\dagger(p_2) \dots a^\dagger(p_N) |0\rangle \quad (15)$$

$$= |p_1, p_2, \dots, p_N\rangle + \int d^3p a^\dagger(p_1) a^\dagger(p) a(p) a^\dagger(p_2) \dots a^\dagger(p_N) |0\rangle \quad (16)$$

The last line uses the fact that $[a^\dagger(p), a^\dagger(p_1)] = 0$ to swap the first two creation operators in the integral. We can now apply the commutator to the terms $a(p) a^\dagger(p_2)$ in the integral on the last line, which yields another term of $|p_1, p_2, \dots, p_N\rangle$ plus another integral with the $a^\dagger(p) a(p)$ operator shifted another slot to the right. The process will continue N times until the $a(p)$ comes up against the $|0\rangle$, so the final integral will be zero because of 1. Thus the final result is

$$\mathcal{N} |p_1, p_2, \dots, p_N\rangle = N |p_1, p_2, \dots, p_N\rangle \quad (17)$$

This argument doesn't actually rely on the momenta of the particles being all different, so we could apply the same reasoning to a general state 8 to see that \mathcal{N} always counts the number of particles in a given state.

We can also work out the commutators of \mathcal{N} with the creation and annihilation operators. We have

$$[\mathcal{N}, a^\dagger(k)] = \int d^3p a^\dagger(p) [a(p), a^\dagger(k)] \quad (18)$$

$$= \int d^3p a^\dagger(p) \delta^3(\mathbf{p} - \mathbf{k}) \quad (19)$$

$$= a^\dagger(k) \quad (20)$$

$$[\mathcal{N}, a(k)] = \int d^3p [a^\dagger(p), a(k)] a(p) \quad (21)$$

$$= - \int d^3p [a(p), a^\dagger(k)] a(p) \quad (22)$$

$$= - \int d^3p \delta^3(\mathbf{p} - \mathbf{k}) a(p) \quad (23)$$

$$= -a(k) \quad (24)$$

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