DIRAC EQUATION: ANGULAR MOMENTUM

Link to: physicspages home page.
To leave a comment or report an error, please use the auxiliary blog.
Post date: 7 June 2018.


The angular momentum operators can be used to generate the infinitesimal transformations

\[ \psi'(x) = \left(1 - \frac{i}{2} J_{\mu\nu}\omega^{\mu\nu}\right) \psi(x) \]  

(1)

where the \( \omega^{\mu\nu} \) are the infinitesimal components of a Lorentz transformation.

Using the notation in L&P’s section 2.4 we have

- \( \psi(x) \) is the function \( \psi \) at spacetime point \( x \).
- \( \psi'(x) \) is the transformation of the function \( \psi \) at the same point \( x \).
- \( \psi'(x') \) is the transformation of the function \( \psi \) at the transformed point \( x' \).

From L&P’s equation 2.41, we have

\[ \psi'(x) - \psi(x) = \psi'(x') - \psi(x) - \partial_\mu \psi'(x) \delta x^\mu \]  

(2)

or

\[ \psi'(x') = \psi'(x) + \partial_\mu \psi'(x) \delta x^\mu \]  

(3)

For an infinitesimal Lorentz transformation

\[ \Lambda_{\mu\nu} = g_{\mu\nu} + \omega_{\mu\nu} \]  

(4)

\[ x'^\mu = \Lambda_{\nu\mu} x^\nu \]  

(5)

\[ = x^\mu + \omega^\mu_{\nu} x^\nu \]  

(6)

\[ \delta x^\mu = x'^\mu - x^\mu \]  

(7)

\[ = \omega^\mu_{\nu} x^\nu \]  

(8)

\[ = \omega^{\mu\nu} x_\nu \]  

(9)

We therefore have

\[ \psi'(x') = \psi'(x) + \omega^{\mu\nu} x_\nu \partial_\mu \psi'(x) \]  

(10)
We can now apply the transformation $\omega^{\mu\nu}$ to this equation, and keep only terms up to first order in $\omega^{\mu\nu}$:

$$\psi'(x') = \left(1 - \frac{i}{2} J_{\mu\nu} \omega^{\mu\nu}\right) \psi(x) + \omega^{\mu\nu} x_\nu \partial_\mu \psi(x)$$  \hspace{1cm} (11)

However, we also know that the LHS, under an infinitesimal Lorentz transformation, has the form

$$\psi'(x') = S(\Lambda) \psi(x)$$  \hspace{1cm} (12)

$$= \left(1 - \frac{i}{4} \sigma_{\mu\nu} \omega^{\mu\nu}\right) \psi(x)$$  \hspace{1cm} (13)

Equating the terms in $\omega^{\mu\nu}$ on both sides, we get

$$\left[ -\frac{i}{2} J_{\mu\nu} + x_\nu \partial_\mu + \frac{i}{4} \sigma_{\mu\nu} \right] \omega^{\mu\nu} = 0$$  \hspace{1cm} (14)

The transformations $\omega^{\mu\nu}$ are arbitrary, but subject to the condition that $\omega^{\mu\nu} = -\omega^{\nu\mu}$. Therefore, we can swap the indexes $\mu \leftrightarrow \nu$ in this equation (and use the antisymmetry of $J_{\mu\nu}$ and $\sigma_{\mu\nu}$) to get

$$\left[ -\frac{i}{2} J_{\nu\mu} + x_\mu \partial_\nu + \frac{i}{4} \sigma_{\nu\mu} \right] \omega^{\nu\mu} = -\left[ -\frac{i}{2} J_{\mu\nu} - x_\mu \partial_\nu + \frac{i}{4} \sigma_{\mu\nu} \right] \omega^{\mu\nu}$$  \hspace{1cm} (15)

Subtracting the RHS from (14) we can now equate the coefficient of $\omega^{\mu\nu}$ to zero to get

$$-i J_{\mu\nu} + x_\nu \partial_\mu - x_\mu \partial_\nu + \frac{i}{2} \sigma_{\mu\nu} = 0$$  \hspace{1cm} (16)

or

$$J_{\mu\nu} = i (x_\mu \partial_\nu - x_\nu \partial_\mu) + \frac{1}{2} \sigma_{\mu\nu}$$  \hspace{1cm} (17)

The first term on the RHS is the traditional orbital angular momentum operator, and the second term represents the spin.

**Pingbacks**

Pingback: Pauli-Lubansky vector and spin in the Dirac equation