DIRAC EQUATION: NONUNIQUENESS OF SOLUTIONS

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Using the Dirac Hamiltonian, the Dirac equation is, in terms of the gamma matrices

\[ i \frac{\partial}{\partial t} \psi(x) = \gamma^0 (-i \gamma \cdot \nabla + m) \psi(x) \]  (1)

Using \((\gamma^0)^2 = 1\), we can multiply by \(\gamma^0\) on the left to get

\[ (i \gamma^\mu \partial_\mu - m) \psi(x) = 0 \]  (2)

The operation of multiplying a vector by \(\gamma^\mu\) and summing is quite common when analyzing the Dirac equation, so a special notation called the *slash notation* is defined as a shorthand. This is

\[ /a \equiv \gamma^\mu a_\mu = \gamma^\mu a^\mu \]  (3)

Using this definition, the Dirac equation takes on the compact form

\[ (i/ - m) \psi(x) = 0 \]  (4)

The gamma matrices are \(4 \times 4\) matrices which satisfy the conditions

\[ \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \]  (5)

\[ (\gamma^\mu)^\dagger = \gamma^0 \gamma^\mu \gamma^0 \]  (6)

If we define an alternative set of gamma matrices by

\[ \tilde{\gamma}^\mu = U \gamma^\mu U^\dagger \]  (7)

where \(U\) is a unitary matrix, so that \(U^\dagger = U^{-1}\), then 5 and 6 are still satisfied. For example
\[ \{ \tilde{\gamma}^\mu, \tilde{\gamma}^\nu \} = U \gamma^\mu U^\dagger U \gamma^\nu U^\dagger + U \gamma^\nu U^\dagger U \gamma^\mu U^\dagger \]  
\[ = U \gamma^\mu \gamma^\nu U^\dagger + U \gamma^\nu \gamma^\mu U^\dagger \]  
\[ = U \{ \gamma^\mu, \gamma^\nu \} U^\dagger \]  
\[ = 2Ug_{\mu\nu}U^\dagger \]  
\[ = 2U \gamma^\mu \gamma^\nu \]  
\[ = 2g_{\mu\nu} \]  

where the penultimate line follows from the fact that \( g_{\mu\nu} \) commutes with any matrix since it is diagonal.

Because the gamma matrices are not unique, the solution \( \psi(x) \) (a column vector with 4 elements) is not unique either. Suppose \( \tilde{\psi}(x) \) satisfies \( 2 \) with \( \gamma^\mu \) replaced by \( \tilde{\gamma}^\mu \). Then

\[ (i\tilde{\gamma}^\mu \partial_\mu - m) \tilde{\psi}(x) = \left( iU \gamma^\mu U^\dagger \partial_\mu - m \right) \tilde{\psi}(x) \]  
\[ = \left( iU \gamma^\mu U^\dagger \partial_\mu - UmU^\dagger \right) \tilde{\psi}(x) \]  
\[ = U \left( i\gamma^\mu \partial_\mu - m \right) U^\dagger \tilde{\psi}(x) \]

This result is equivalent to \( 2 \) if \( \tilde{\psi}(x) = U\psi(x) \).

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