DIRAC EQUATION: LINEAR INDEPENDENCE OF MATRICES

In the Dirac equation, we introduced five $\gamma$ matrices labelled $\gamma^\mu$ and $\gamma^5$. These matrices satisfied the properties

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$$  \hspace{1cm} (1)

$$\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}$$  \hspace{1cm} (2)

$$(\gamma^0)^2 = +1$$  \hspace{1cm} (3)

$$(\gamma^i)^2 = -1$$  \hspace{1cm} (4)

$$(\gamma^5)^2 = +1$$  \hspace{1cm} (5)

$$\text{Tr}\ \gamma^\mu = 0$$  \hspace{1cm} (6)

$$\text{Tr}\ \gamma^5 = 0$$  \hspace{1cm} (7)

$$\{\gamma^\mu,\gamma^5\} = 0$$  \hspace{1cm} (8)

We also introduced the matrices

$$\sigma^{\mu\nu} \equiv \frac{i}{2} [\gamma^\mu,\gamma^\nu] = -\sigma^{\nu\mu}$$  \hspace{1cm} (9)

We defined the set of 16 matrices defined as

$$\Gamma = \{1, \gamma^\mu, \sigma^{\mu\nu}, \gamma^5\gamma^\mu, \gamma^5\}$$  \hspace{1cm} (10)

and showed that the product of any number of $\gamma^\mu$s can be reduced to a linear combination of matrices $\Gamma_r$ from this set. We’d now like to show that the matrices in this set are linearly independent, that is, if we form the linear combination

$$\sum_{r=1}^{16} a_r \Gamma_r = 0$$  \hspace{1cm} (11)
then all coefficients $a_r$ must be zero. The method of doing this is to multiply successively by each of the $\Gamma_r$ and then take the trace. In order to use this method, we need the traces of each of the $\Gamma_r$. We already know from $6$ and $7$ that the traces of all single $\gamma^\mu$s and $\gamma^5$ are zero. Since we’re dealing with $4 \times 4$ matrices, the trace of the unit matrix $1$ is $4$.

Using the cyclic property of the trace, we have

$$\text{Tr} \sigma^{\mu\nu} = \text{Tr} \left( \gamma^\mu \gamma^\nu \right) - \text{Tr} \left( \gamma^\nu \gamma^\mu \right)$$  \hspace{1cm} (12)

$$= \text{Tr} \left( \gamma^\mu \gamma^\nu \right) - \text{Tr} \left( \gamma^\nu \gamma^\mu \right)$$  \hspace{1cm} (13)

$$= 0$$  \hspace{1cm} (14)

Combining the cyclic property with $8$, we see that

$$\text{Tr} \left( \gamma^5 \gamma^\mu \right) = \text{Tr} \left( \gamma^\mu \gamma^5 \right)$$  \hspace{1cm} (15)

$$= -\text{Tr} \left( \gamma^\mu \gamma^5 \right)$$  \hspace{1cm} (16)

$$= 0$$  \hspace{1cm} (17)

Therefore, the traces of all the $\Gamma_r$ except the unit matrix are zero.

Now we need to consider the squares of each of the $\Gamma_r$. We saw earlier that

$$\sigma^{\mu\nu} = i\gamma^\mu \gamma^\nu$$  \hspace{1cm} (18)

Therefore, if $\mu \neq \nu$

$$\left( \sigma^{\mu\nu} \right)^2 = -\gamma^\mu \gamma^\nu \gamma^\mu \gamma^\nu$$  \hspace{1cm} (19)

$$= (\gamma^\mu)^2 (\gamma^\nu)^2$$  \hspace{1cm} (20)

$$= \pm 1$$  \hspace{1cm} (21)

where the sign depends on precise values of $\mu$ and $\nu$, using $3$ and $4$.

Also, using $8$ and $5$:

$$\left( \gamma^5 \gamma^\mu \right)^2 = \gamma^5 \gamma^\mu \gamma^5 \gamma^\mu$$  \hspace{1cm} (22)

$$= -\left( \gamma^5 \right)^2 (\gamma^\mu)^2$$  \hspace{1cm} (23)

$$= \pm 1$$  \hspace{1cm} (24)

Thus combining these results with $3$, $4$ and $5$, the square of every $\Gamma_r$ is $\pm 1$, that is, a multiple of the unit matrix.
Now let’s return to the original statement that we’re trying to prove. Choose one of the members, say $\Gamma_s$, of the set, and multiply the sum by $\Gamma_s$ so that we get a new sum:

$$\Gamma_s \sum_{r=1}^{16} a_r \Gamma_r = \sum_{r=1}^{16} a_r (\Gamma_s \Gamma_r) = 0 \quad (25)$$

Every term in the resulting product consists of a product of two of the $\Gamma_r$, and we know that any product of two $\Gamma_r$ can be written as a linear combination of the original $\Gamma_r$. One of these terms will consist of $\Gamma_s^2$, which we’ve seen is a multiple of the unit matrix. The other terms will consist of other members of the set $\Gamma$, all of which have zero trace. If we take the trace of the sum $25$, this is the sum of the traces of each term in the sum, and the only term that has a non-zero trace is $a_s \Gamma_s^2$, so the coefficient of this term must be $a_s = 0$ to make the total trace equal to zero. This argument applies to each of the $\Gamma_r$ in turn, so in order for $25$ and hence $11$ to be true is if all the $a_r = 0$. QED.

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