GAMMA MATRICES: CONTRACTIONS

Here are a few formulas involving contractions of the gamma matrices. We make use of the anticommutation relation

\[ \{ \gamma_\mu, \gamma_\nu \} = 2g_{\mu\nu} \]  

and the squares

\[ (\gamma_0)^2 = +1 \]  
\[ (\gamma_i)^2 = -1 \]  

Note that for \( i = 1, 2, 3 \) (no sum over \( i \)):

\[ \gamma_i \gamma^i = \gamma_0 g^{ii} \gamma_i = (\gamma_i)^2 g^{ii} = +1 \]  

The simples contraction is

\[ \gamma_\lambda \gamma^\lambda = (\gamma_0)^2 + \sum_{i=1}^{3} \gamma_i \gamma^i = 4 \]  

Next, we have

\[ \gamma_\lambda \gamma_\mu \gamma^\lambda = (2g_{\lambda\mu} - \gamma_\mu \gamma_\lambda) \gamma^\lambda \]  
\[ = 2\gamma_\mu - 4\gamma_\mu \]  
\[ = -2\gamma_\mu \]  

Next:
\[ \gamma_\lambda \gamma_\mu \gamma_\nu \gamma_\rho = g_\lambda^\rho \gamma_\lambda \gamma_\mu \gamma_\nu \gamma_\rho \quad (9) \]
\[ = g_\lambda^\rho \left( \gamma_\lambda \gamma_\mu \left( 2g_{\nu\rho} - \gamma_\rho \gamma_\nu \right) \right) \quad (10) \]
\[ = 2\gamma_\nu \gamma_\mu - \left( \gamma_\lambda \gamma_\mu \gamma_\lambda \right) \gamma_\nu \quad (11) \]
\[ = 2\gamma_\nu \gamma_\mu + 2\gamma_\mu \gamma_\nu \quad (12) \]
\[ = 2\gamma_\nu \gamma_\mu + 2 \left( 2g_{\mu\nu} - \gamma_\mu \gamma_\nu \right) \quad (13) \]
\[ = 4g_{\mu\nu} \quad (14) \]

Finally:

\[ \gamma_\lambda \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\lambda = g_\lambda^\alpha \gamma_\lambda \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\alpha \quad (15) \]
\[ = g_\lambda^\alpha \left( \gamma_\lambda \gamma_\mu \gamma_\nu \left( 2g_{\rho\alpha} - \gamma_\alpha \gamma_\rho \right) \right) \quad (16) \]
\[ = 2\gamma_\rho \gamma_\mu \gamma_\nu - \left( \gamma_\lambda \gamma_\mu \gamma_\nu \gamma_\lambda \right) \gamma_\rho \quad (17) \]
\[ = 2\gamma_\rho \gamma_\mu \gamma_\nu - 4g_{\mu\rho} \gamma_\nu \quad (18) \]
\[ = 2\gamma_\rho \left( 2g_{\mu\nu} - \gamma_\nu \gamma_\mu \right) - 4g_{\mu\nu} \gamma_\rho \quad (19) \]
\[ = -2\gamma_\rho \gamma_\nu \gamma_\mu \quad (20) \]

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