DIRAC SPINORS: GORDON IDENTITY

The Dirac spinors satisfy the relations

\begin{align}
(p - m) u_s(p) &= 0 \\
(p + m) v_s(p) &= 0 \\
\overline{u}_s(p)(p - m) &= 0 \\
\overline{v}_s(p)(p + m) &= 0
\end{align}

where a barred spinor is defined by

\[ \overline{\psi} \equiv \psi^\dagger \gamma^0 \]

We’ve seen that these spinors satisfy the identities:

\begin{align}
\overline{u}_s(p) p^\mu u_r(p) &= m \overline{u}_s(p) \gamma^\mu u_r(p) \\
\overline{v}_s(p) p^\mu v_r(p) &= -m \overline{v}_s(p) \gamma^\mu v_r(p)
\end{align}

The Gordon identity relates spinors at different momenta. We can derive it using the properties of the \text{gamma matrices}. To streamline the notation, I’ll use the following definitions:

\begin{align}
u &\equiv u_s(p) \\
u' &\equiv \overline{u}_s(p')
\end{align}

with a similar notation for \(v\) and \(v'\). The subscript \(s\) can be 1 or 2. Multiply \(\gamma^\mu\) on the left by \(\overline{u}' \gamma^\mu\):

\[ \overline{u}' \gamma^\mu \gamma^\alpha p_\alpha u = m \overline{u}' \gamma^\mu u \]

Now multiply \(\gamma^\mu\) (with momentum \(p'\)) on the right by \(\gamma^\mu u\):

\[ \overline{u}' \gamma^\alpha p'_\alpha \gamma^\mu u = m \overline{u}' \gamma^\mu u \]
Now add these two equations:
\[ \mp' \left( \gamma^\mu \gamma^\alpha p_\alpha + \gamma^\alpha \gamma'^\mu p'_\alpha \right) u = 2m\mp' \gamma^\mu u \]  
(12)

Using the anticommutator
\[ \{\gamma^\mu, \gamma^\alpha\} = 2g^{\mu\alpha} \]  
(13)

Now consider (using the definition of \( \sigma^{\mu\alpha} \)):
\[ -i\sigma^{\mu\alpha} q_\alpha = -i \frac{i}{2} (\gamma^\mu \gamma^\alpha - \gamma^\alpha \gamma^\mu) (p_\alpha - p'_\alpha) \]  
(14)
\[ = \frac{1}{2} (\gamma^\mu \gamma^\alpha - 2g^{\alpha\mu} + \gamma^\mu \gamma^\alpha) p_\alpha + \frac{1}{2} (\gamma^\alpha \gamma'^\mu - 2g^{\mu\alpha} + \gamma^\alpha \gamma'^\mu) p'_\alpha \]  
(15)
\[ = \gamma^\mu \gamma^\alpha p_\alpha + \gamma^\alpha \gamma'^\mu p'_\alpha - (p + p')^\mu \]  
(16)

The first two terms are the same as the quantity in parentheses in (12) so, substituting this back into (12) we have
\[ \mp' \gamma^\mu u = \frac{1}{2m} \mp' \left[ (p + p')^\mu - i\sigma^{\mu\alpha} q_\alpha \right] u \]  
(18)

We can get a similar expression involving \( v \) by replacing \( m \) by \(-m\):
\[ \mp' \gamma^\mu v = -\frac{1}{2m} \mp' \left[ (p + p')^\mu - i\sigma^{\mu\alpha} q_\alpha \right] v \]  
(19)

This is the Gordon identity.