EXPLICIT SOLUTIONS OF DIRAC EQUATION IN
DIRAC-PAULI REPRESENTATION

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Post date: 14 June 2018.
References: Amitabha Lahiri & P. B. Pal, A First Book of Quantum Field
Theory, Second Edition (Alpha Science International, 2004) - Chapter 4,
Problem 4.11.
The solution of the Dirac equation for a free particle is

\[ \psi(x) \sim \begin{cases} 
    u_s(p) e^{-ip \cdot x} \\
    v_s(p) e^{ip \cdot x} 
\end{cases} \]  

(1)

where \( u_s \) and \( v_s \) are 4-component spinors and \( s = + \) or \( - \). These spinors satisfy

\[ (\gamma^0 - m) u_s(p) = 0 \]  

(2)

\[ (\gamma^0 + m) v_s(p) = 0 \]  

(3)

To find explicit forms for the spinors, we need an explicit representation
of the gamma matrices. One such representation is the Dirac-Pauli repre-
sentation in which the matrices are given by

\[ \gamma^0 = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \]  

(4)

\[ \gamma^i = \begin{bmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{bmatrix} \]  

(5)

where the \( \sigma^i \) are the Pauli matrices

\[ \sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \quad \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}; \quad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \]  

(6)

Each of the entries in (4) and (5) is a \( 2 \times 2 \) matrix, while the entries in the
Pauli matrices are ordinary numbers. The four-vector \( p \) has the general form

\[ p = (E_p, p) \]  

(7)

where
\[ E_p = \sqrt{p^2 + m^2} \]  

(8)

With the particular representation of the gamma matrices, we have

\[ \gamma_0 E_p - \gamma \cdot p - mI = \left[ \begin{array}{cc} E_p - m & -\sigma \cdot p \\ \sigma \cdot p & -E_p - m \end{array} \right] \]  

(9)

(10)

Plugging this into (8) results in

\[ (E_p - m) \phi_t - \sigma \cdot p \phi_b = 0 \]  

(11)

\[ \sigma \cdot p \phi_t - (E_p + m) \phi_b = 0 \]  

(12)

where \( \phi_t \) is a column vector with the top two components of \( u_s \) and \( \phi_b \) contains the bottom two components of \( u_s \). Each term in the equations \( \Pi \) is a two-component vector, since it is the sum of terms containing the product of a \( 2 \times 2 \) matrix and a 2-component column vector.

To find \( u_s \), we assume that everything in \( \Pi \) is specified except for \( \phi_t \) and \( \phi_b \), so these two equations comprise a system of two equations in two unknowns. This system has a solution if the determinant of the coefficients is zero, which we can verify.

\[
\begin{vmatrix}
E_p - m & -\sigma \cdot p \\
\sigma \cdot p & -E_p - m
\end{vmatrix} = -(E_p - m)(E_p + m) + (\sigma \cdot p)^2
\]  

(13)

\[
= -E_p^2 + m^2 + (\sigma \cdot p)^2
\]  

(14)

\[
= -p^2 I + (\sigma \cdot p)^2
\]  

(15)

The last term can be worked out using (6):

\[
(\sigma \cdot p)^2 = \begin{bmatrix}
p^3 & p^1 - ip^2 \\
p^1 + ip^2 & -p^3
\end{bmatrix}^2
\]  

(16)

\[
= \begin{bmatrix}
p^2 & 0 \\
0 & p^2
\end{bmatrix}
\]  

(17)

\[
= p^2 I
\]  

(18)

Plugging this into (15) we have

\[
\begin{vmatrix}
E_p - m & -\sigma \cdot p \\
\sigma \cdot p & -E_p - m
\end{vmatrix} = 0
\]  

(19)

as required.
From the second of [11] we have

$$\phi_b = \frac{\sigma \cdot p}{E_p + m} \phi_t$$  \hspace{1cm} (20)

so the solution is

$$u_{\pm}(p) = A \left[ \begin{array}{c} \chi_{\pm} \\ \frac{\sigma \cdot p}{E_p + m} \chi_{\pm} \end{array} \right]$$  \hspace{1cm} (21)

where $A$ is a constant determined by the normalization condition

$$u_{r}^\dagger(p) u_{s}(p) = v_{r}^\dagger(p) v_{s}(p) = 2E_p \delta_{rs}$$  \hspace{1cm} (22)

The 2-component vectors $\chi_{\pm}$ can in fact be any pair of linearly independent, normalized vectors. In this case, the normalization constant $A$ is

$$A = \sqrt{E_p + m}$$  \hspace{1cm} (23)

For the particular solution in L&P they choose

$$\chi_{+} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$  \hspace{1cm} (24)

$$\chi_{-} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$  \hspace{1cm} (25)

which gives

$$u_{\pm}(p) = \sqrt{E_p + m} \left[ \begin{array}{c} \chi_{\pm} \\ \frac{\sigma \cdot p}{E_p + m} \chi_{\pm} \end{array} \right]$$  \hspace{1cm} (26)

We can go through a similar argument to find $v_{\pm}$, and with the same choice of $\chi_{\pm}$, we have

$$v_{\pm}(p) = \pm \sqrt{E_p + m} \left[ \begin{array}{c} \frac{\sigma \cdot p}{E_p + m} \chi_{\pm} \\ \chi_{\mp} \end{array} \right]$$  \hspace{1cm} (27)

The swapping of $\chi_{+}$ with $\chi_{-}$ is just a convention.

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