NORMALIZING BOSON AND FERMION STATES

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We now consider the dimensions of the boson and fermion states. The scalar field operator in 4-dimensional spacetime has the dimensions of mass, so we can infer the dimension of the creation and annihilation operators from the Fourier expansion of the field

$$\phi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{\sqrt{2E_p}} \left( a(p)e^{-ip\cdot x} + a^\dagger(p)e^{ip\cdot x} \right)$$  (1)

In natural units, momentum and energy both have units of mass, so the units of \( \frac{d^3p}{\sqrt{2E_p}} \) are \([\text{mass}^3] [\text{mass}^{-1/2}] = [\text{mass}^{5/2}]\). To make the integrand have units of \([\text{mass}]\), the creation and annihilation operators must each have units of \([\text{mass}^{-3/2}]\).

We begin the construction of Fock space with the vacuum state |0\>. We take this state to be normalized so that

$$\langle 0 | 0 \rangle = 1$$  (2)

which implies that |0\> must be dimensionless. However, we originally defined the state with a single particle as

$$|p\rangle = a^\dagger (p) |0\rangle$$  (3)

As a result of the units of \( a^\dagger \), this gives |p\> the dimensions of \([\text{mass}^{-3/2}]\). Adding more particles by applying \( a^\dagger \) again gives states with units of successive factors of \([\text{mass}^{-3/2}]\), so we end up with a Fock space in which the dimensions of the states vary according to the numbers of particles in the states.

Also, because of the orthogonality condition

$$\langle p' | p \rangle = \delta^3 (p' - p)$$  (4)
we find that a one-particle state is normalized to infinity:

\[ \langle p | p \rangle = \delta^3 (0) = \infty \quad (5) \]

If we want all states to be dimensionless and also to overcome the infinite normalization problem, we can confine the states to a finite volume \( V \) and introduce a numerical normalization factor. L&P explain how this is done in their equations 6.12 to 6.18. For a scalar field, a one-particle state is defined as

\[ |B(p)\rangle \equiv \sqrt{\frac{(2\pi)^3}{V}} a^\dagger (p) |0\rangle \quad (6) \]

For electrons and positrons, we have

\[ |e^-(p, s)\rangle \equiv \sqrt{\frac{(2\pi)^3}{V}} f^\dagger_s (p) |0\rangle \quad (7) \]

\[ |e^+(p, s)\rangle \equiv \sqrt{\frac{(2\pi)^3}{V}} \hat{f}^\dagger_s (p) |0\rangle \quad (8) \]

Since length has the dimension of \([\text{mass}^{-1}]\) in natural units, the factor of \(1/\sqrt{V}\) contributes a dimension of \([\text{mass}^{3/2}]\) which cancels the \([\text{mass}^{-3/2}]\) of the creation operators. The factor of \((2\pi)^3\) is introduced for convenience in later calculations.

Although this solves the problem of dimensionality, the normalization problem is still there. For example, using the [commutation relation] for scalar operators

\[
\langle B(p') | B(p) \rangle = \frac{(2\pi)^3}{V} \langle 0 | a(p') a^\dagger (p) |0\rangle \quad (9)
\]

\[ = \frac{(2\pi)^3}{V} \langle 0 | \delta^3 (p' - p) + a^\dagger (p) a (p') |0\rangle \quad (10) \]

\[ = \frac{(2\pi)^3}{V} \langle 0 | \delta^3 (p' - p) |0\rangle \quad (11) \]

\[ = \frac{(2\pi)^3}{V} \delta^3 (p' - p) \quad (12) \]

where in the second line we used \(a (p') |0\rangle = 0\). Similar relations apply for fermion operators. Thus we still have
\[ \langle B(p)|B(p) \rangle = \frac{(2\pi)^3}{V} \delta^3(0) = \infty \] (13)

The problem is ‘solved’ by restricting the system to a box of volume \( V \) and defining the delta function as

\[ \delta^3(p) = \lim_{V \to \infty} \left( \frac{1}{(2\pi)^3} \int_V d^3x \ e^{-ip \cdot x} \right) \] (14)

Then for finite volume we have

\[ \delta^3(0) = \lim_{V \to \infty} \left( \frac{V}{(2\pi)^3} \right) \] (15)

so if we insert this into (13) before taking the limit of infinite volume, then the volume factors cancel and we end up with

\[ \langle B(p)|B(p) \rangle = \lim_{V \to \infty} \left( \frac{(2\pi)^3}{V} \frac{V}{(2\pi)^3} \right) = 1 \] (16)

As with all calculations involving delta functions, this makes me feel a bit uneasy, but it seems to work. Using this technique, we find that all states in Fock space are now normalized to unity.

We can now calculate the action of the field operators on one-particle states. For scalar fields, we can write

\[ \phi(x) = \phi_+(x) + \phi_-(x) \] (17)

where \( \phi_+ \) contains the annihilation operators and \( \phi_- \) contains the creation operators:

\[ \phi_+(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{\sqrt{2E_p}} a(p) e^{-ip \cdot x} \] (18)

\[ \phi_-(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{\sqrt{2E_p}} a_\dagger (p) e^{ip \cdot x} \] (19)

To calculate the effect of \( \phi_+ (x) \) on \( |B(k)\rangle \) we have
\[ \phi_+ (x) |B (k)\rangle = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3 p}{\sqrt{2E_p}} a (p) e^{-ip \cdot x} \sqrt{\frac{(2\pi)^3}{V}} a^\dagger (k) |0\rangle \] 

\[ = \frac{1}{\sqrt{2V}} \int \frac{d^3 p}{\sqrt{E_p}} e^{-ip \cdot x} a (p) a^\dagger (k) |0\rangle \] 

\[ = \frac{1}{\sqrt{2V}} \int \frac{d^3 p}{\sqrt{E_p}} e^{-ip \cdot x} a (p) a^\dagger (k) |0\rangle \] 

\[ = \frac{1}{\sqrt{2V}} \int \frac{d^3 p}{\sqrt{E_p}} e^{-ip \cdot x} \left( \delta^3 (p - k) + a^\dagger (k) a (p) \right) |0\rangle \] 

\[ = \frac{1}{\sqrt{2V}} \int \frac{d^3 p}{\sqrt{E_p}} e^{-ip \cdot x} \delta^3 (p - k) |0\rangle \] 

\[ = \frac{1}{\sqrt{2E_k V}} e^{-ik \cdot x} |0\rangle \] 

Note that the delta function also forces \( E_p = \sqrt{p^2 + m^2} \) to become \( E_k \) when \( p \) becomes \( k \).

A similar relation holds for fermion fields. The Fourier decomposition is

\[ \psi_+ (x) = \int \frac{d^3 p}{\sqrt{2 (2\pi)^3 E_p}} \sum_{s=1,2} \left( f_s (p) u_s (p) e^{-ip \cdot x} + f_s^\dagger (p) v_s (p) e^{ip \cdot x} \right) \] 

so we have

\[ \psi_+ (x) = \int \frac{d^3 p}{\sqrt{2 (2\pi)^3 E_p}} \sum_{s=1,2} f_s (p) u_s (p) e^{-ip \cdot x} \] 

Operating on an electron state \( \rangle \), we have, using the anticommutation relation
\[ \psi_+(x) |e^-(k, s)\rangle = \int \frac{d^3 p}{\sqrt{2} (2\pi)^3 E_p} \sum_{r=1,2} f_r(p) u_r(p) e^{-ip \cdot x} \sqrt{\frac{(2\pi)^3}{V}} f_s^\dagger(k) |0\rangle \]

\[ = \frac{1}{\sqrt{2V}} \int \frac{d^3 p}{\sqrt{E_p}} e^{-ip \cdot x} \sum_{r=1,2} u_r(p) f_r(p) f_s^\dagger(k) |0\rangle \]

\[ = \frac{1}{\sqrt{2V}} \int \frac{d^3 p}{\sqrt{E_p}} e^{-ip \cdot x} \sum_{r=1,2} u_r(p) \left[ \delta_{rs} \delta^3(p - k) - f_s^\dagger(k) f_r(p) \right] |0\rangle \]

\[ = \frac{1}{\sqrt{2E_k V}} e^{-ik \cdot x} u_s(k) |0\rangle \]

We can do a similar calculation for \( \overline{\psi}_+ \) to get

\[ \overline{\psi}_+(x) |e^+(k, s)\rangle = \frac{1}{\sqrt{2E_k V}} e^{-ik \cdot x} \overline{u}_s(k) |0\rangle \]

For reference, we also list the adjoints:

\[ \langle B(k) | \phi_-(x) = \frac{1}{\sqrt{2E_k V}} e^{ik \cdot x} \langle 0 | \]

\[ \langle e^-(k, s) | \overline{\psi}_-(x) = \frac{1}{\sqrt{2E_k V}} e^{ik \cdot x} \overline{u}_s(k) \langle 0 | \]

\[ \langle e^+(k, s) | \psi_-(x) = \frac{1}{\sqrt{2E_k V}} e^{ik \cdot x} u_s(k) \langle 0 | \]

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