DECAY RATES

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References: Amitabha Lahiri & P. B. Pal, *A First Book of Quantum Field Theory*, Second Edition (Alpha Science International, 2004) - Chapter 7.

It's worth reviewing the formula for the calculation of a decay rate in quantum field theory. The general form of an S-matrix element is

$$S_{fi}^{(j)} = i\,(2\pi)^4\,\delta^4\left(\sum_i p_i - \sum_f p_f\right)\prod_i \frac{1}{\sqrt{2E_iV}}\prod_f \frac{1}{\sqrt{2E_fV}}\mathscr{M}_{fi}^{(i)} \tag{1}$$

where the superscript (i) indicates the ith order term in the Wick expansion and $\mathscr{M}_{fi}^{(i)}$ is the corresponding Feynman amplitude. The suffix fi indicates a transition from an initial to a final state. The actual probability of the transition ever occurring (that is, over all times) is the square modulus of $S_{fi}^{(i)}$, but this leads to a couple of problems. The first is that we need to square the delta function. L&P cope with this problem by starting with the observation that, for an arbitrary function $f(p)$

$$\delta^4(p)\,f(p) = \delta^4(p)\,f(0) \tag{2}$$

since the delta function is zero everywhere except at $p=0$. If $f(p)$ is itself another delta function, we can apply the same logic to get

$$\left[\delta^4(p)\right]^2 = \delta^4(p)\,\delta^4(0) \tag{3}$$

This might not appear to help much, as we're still multiplying together two infinities at the point $p=0$. L&P get around this by defining the delta function as the limit of an integral over a finite volume, in their equations 6.16 and 6.17 (for a 3-dimensional delta function). For the 4-dimensional form, we have

$$\delta^4(p) = \lim_{V\to\infty}\lim_{T\to\infty}\left[\frac{1}{(2\pi)^4}\int_{V,T} d^4x\; e^{-ip\cdot x}\right] \tag{4}$$

The idea is that if we choose the volume and time of integration such that an integral number of oscillations of the exponential fit into the boundaries, then the integral will be zero, except when $p=0$, in which case we have, before taking the limits:

$$\delta^4(0)_p = \frac{1}{(2\pi)^4}\int_{V,T} d^4x = \frac{VT}{(2\pi)^4} \tag{5}$$

As this result goes to infinity in the limits, the delta function satisfies the conditions of being zero everywhere except at $p=0$, where it is infinite.

With this definition, we can write the square of 1 as

$$\left|S_{fi}^{(i)}\right|^2 = \frac{VT}{(2\pi)^4}(2\pi)^8\delta^4\left(\sum_i p_i - \sum_f p_f\right)\prod_i \frac{1}{2E_iV}\prod_f \frac{1}{2E_fV}\left|\mathscr{M}_{fi}^{(i)}\right|^2 \tag{6}$$

Note that we have not yet taken the limits $V\to\infty$ or $T\to\infty$.

The probability per unit time is therefore

$$\frac{\left|S_{fi}^{(i)}\right|^2}{T} = V(2\pi)^4\delta^4\left(\sum_i p_i - \sum_f p_f\right)\prod_i \frac{1}{2E_iV}\prod_f \frac{1}{2E_fV}\left|\mathscr{M}_{fi}^{(i)}\right|^2 \tag{7}$$

This formula still specifies that the initial and final momenta are exactly known in the delta function. In the infinite volume limit, the momenta can take on continuous values. We now divide up phase space (the product of position and momentum spaces) into discrete cells, each of which can house a single state of the system. The argument is essentially the same as the one we met when discussing the number of states of an ideal gas molecule. Because of the uncertainty principle, the volumes of space and momentum space that can be occupied satisfy (roughly) $\Delta x\Delta p\approx h$ in each dimension, or $(\Delta x\Delta p)^3\approx h^3$ in 3 dimensions. Thus the number of states in a spatial volume V and momentum volume d^3p is about

$$N\approx\frac{Vd^3p}{h^3}=\frac{Vd^3p}{(2\pi\hbar)^3}=\frac{Vd^3p}{(2\pi)^3} \tag{8}$$

where the last equality uses natural units where $\hbar=1$. We get one factor like this for each final state particle.

Returning to 7, to get the total decay rate Γ, we need to multiply the RHS by 8 for each final state momentum and then integrate over all final state momenta. In a decay process, there is a single initial particle, and it is normal for the initial state momentum to be precisely known (for example, the decaying particle could be at rest, or it could be in a beam of particles all travelling with the same momentum). We therefore have

$$\Gamma = \int \prod_f \frac{V d^3 p_f}{(2\pi)^3} \frac{\left|S_{fi}^{(i)}\right|^2}{T} \tag{9}$$

$$= \frac{1}{2E_i} \int \prod_f \left[\frac{V d^3 p_f}{(2\pi)^3 2E_f V}\right] (2\pi)^4 \delta^4 \left(p_i - \sum_f p_f\right) \left|\mathscr{M}_{fi}^{(i)}\right|^2 \tag{10}$$

$$= \frac{1}{2E_i} \int \prod_f \left[\frac{d^3 p_f}{(2\pi)^3 2E_f}\right] (2\pi)^4 \delta^4 \left(p_i - \sum_f p_f\right) \left|\mathscr{M}_{fi}^{(i)}\right|^2 \tag{11}$$

Note that the volume has conveniently cancelled out of the final result, so we don't need to take the limit of infinite volume. This is L&P's equation 7.8 which is the basis for all decay rate calculations.

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