BOSON DECAY RATE: SUM OVER SPINS

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In the calculation of the decay rate of a boson via the Yukawa interaction, we come across the term

\[ \Sigma_{\text{spin}} = \sum_{s,s'} |\overline{u}_s (p) v_{s'} (p')|^2 \]

(1)

\[ = \sum_{s,s'} [\overline{u}_s (p) v_{s'} (p')][\overline{u}_s (p) v_{s'} (p')]^* \]

(2)

L&P work out the complex conjugate term in their equation 7.13, but we can work out a more general case as follows. For a general $4 \times 4$ matrix $F$ we wish to find $[\pi F v]^*$. I’ve suppressed the dependence of $u$ and $v$ on spin and momentum to make the notation easier. First, note that since $\pi$ is a $1 \times 4$ matrix (a row vector), $F$ is a $4 \times 4$ matrix and $v$ is a $4 \times 1$ matrix (a column vector), the product $\pi F v$ is just a single number (or a $1 \times 1$ matrix if you prefer), so its complex conjugate is also its hermitian conjugate. Therefore

\[ [\overline{u} F v]^* = [\overline{u} F v]^\dagger \]

(3)

\[ = \left[ u^\dagger \gamma_0 F v \right]^\dagger \]

(4)

\[ = v^\dagger F^\dagger \gamma_0 u \]

(5)

\[ = v^\dagger \gamma_0 \gamma_0 F^\dagger \gamma_0 u \]

(6)

\[ = \overline{v} F^\dagger u \]

(7)

where

\[ F^\dagger \equiv \gamma_0 F^\dagger \gamma_0 \]

(8)

and we’ve used the properties of $\gamma_0$: $\gamma_0^\dagger = \gamma_0$ and $(\gamma_0)^2 = I$.

When we use this result in (2) we have

When we multiply a $4 \times 1$ matrix into a $1 \times 4$ matrix, the result is a $4 \times 4$ matrix.
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\[ \Sigma_{\text{spin}} = \sum_{s,s'} \left[ (\overline{u}_s(p))_\alpha (v_{s'}(p'))_\beta \right] \left[ (\overline{v}_{s'}(p'))_\beta (u_s(p))_\alpha \right] \]  
\[ = \left[ \sum_s u_s(p) \overline{u}_s(p) \right] \left[ \sum_{s'} v_{s'}(p') \overline{v}_{s'}(p') \right] \]  

(9)

where the subscripts \(\alpha\) and \(\beta\) refer to the components of the matrices.

Again, we can work out this term for the more general case.

\[ \sum_{s,s'} |u_s(p) F v_{s'}(p')|^2 = \sum_{s,s'} \left[ (\overline{u}_s(p))_\alpha F_{\alpha\beta} (v_{s'}(p'))_\beta \right] \left[ (\overline{v}_{s'}(p'))_\beta F^\dagger_{\delta\epsilon} (u_s(p))_\epsilon \right] \]

(11)

\[ = \left[ \sum_s u_s(p) \overline{u}_s(p) \right] F_{\alpha\beta} \left[ \sum_{s'} v_{s'}(p') \overline{v}_{s'}(p') \right] F^\dagger_{\delta\epsilon} \]  

(12)

We can now use the results

\[ \sum_s u_s(p) \overline{u}_s(p) = \not{p} + m \]  
(13)

\[ \sum_s v_s(p) \overline{v}_s(p) = \not{p} - m \]  
(14)

We get

\[ \sum_{s,s'} |u_s(p) F v_{s'}(p')|^2 = (\not{p} + m) F_{\alpha\beta} (\not{p}' - m) F^\dagger_{\delta\epsilon} \]  

(15)

Note that summing over the Greek subscripts means that the four terms are just a matrix product, with one extra sum over the first and last indexes. That is

\[ (\not{p} + m) F_{\alpha\beta} (\not{p}' - m) F^\dagger_{\delta\epsilon} = \sum_\epsilon \left[ (\not{p} + m) F (\not{p}' - m) F^\dagger \right]_{\epsilon\epsilon} \]  

(16)

\[ = \text{Tr} \left[ (\not{p} + m) F (\not{p}' - m) F^\dagger \right] \]  

(17)

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