PHOTON FIELD MOMENTUM

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Greiner & Reinhardt, *Field Quantization*, Chapter 7.

In their section 8.3, L&P show that the Lagrangian for the photon field can be rewritten as

\[
\mathcal{L} = -\frac{1}{2} (\partial_{\mu} A_{\nu}) (\partial^{\mu} A^{\nu}) + \frac{1}{2} \left(1 - \frac{1}{\xi}\right) (\partial_{\mu} A^{\mu})^2
\]

where the last term incorporates the gauge fixing term

\[
\mathcal{L}_{\text{GF}} = -\frac{1}{2\xi} (\partial_{\mu} A^{\mu})^2
\]

The field operators \( A^{\mu} \) can be written as Fourier decompositions

\[
A^{\mu}(x) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}} \sum_{r=0}^{3} \left[ \epsilon^{\mu}_{r}(k) a_{r}(k) e^{-ik \cdot x} + \epsilon^{*\mu}_{r}(k) a^{\dagger}_{r}(k) e^{ik \cdot x} \right]
\]

We now wish to find the components of the four-momentum \( P^{\mu} \) in terms of creation and annihilation operators. The four-momentum is defined in terms of the stress-energy tensor by L&P’s eqn 2.55:

\[
P^{\mu} = \int d^3x \ T^{0\mu}
\]

where

\[
T^{0\mu} = \frac{\partial \mathcal{L}}{\partial A^{\rho}} \partial^{\mu} A^{\rho} - g^{0\mu} \mathcal{L}
\]

From \( \Box \) we have

\[
\frac{\partial \mathcal{L}}{\partial A^{\rho}} = -\dot{A}^{\rho} + \left(1 - \frac{1}{\xi}\right) g^{\rho 0} \partial_{\nu} A^{\nu}
\]
Let’s consider the spatial components $P^j$ first (where $j = 1, 2, 3$). From \[5\] and \[6\] we have

$$T^{0j} = \frac{\partial \mathcal{L}}{\partial \dot{A}^\rho} \partial^j A^\rho$$

$$= -\dot{A}_\rho \partial^j A^\rho + \left( 1 - \frac{1}{\xi} \right) (\partial_\nu A^\nu) \partial^j A^0$$ \quad (7)

We are interested in terms that give physically meaningful results, and for this we need to look at the Gupta-Bleuler condition (L&P section 8.6) which states that for any two physical states $|\Psi\rangle$ and $|\Psi'\rangle$, we must have

$$\langle \Psi' | \partial_\nu A^\nu | \Psi \rangle = 0$$ \quad (9)

Taking the matrix element of the second term in \[8\] we can insert a complete set of states to get

$$\left( 1 - \frac{1}{\xi} \right) \langle \Psi' | (\partial_\nu A^\nu) \partial^j A^0 | \Psi \rangle = \left( 1 - \frac{1}{\xi} \right) \int dk \langle \Psi' | \partial_\nu A^\nu | k \rangle \langle k | \partial^j A^0 | \Psi \rangle$$

$$= 0$$ \quad (10)

where the conclusion follows because we must have $\langle \Psi' | \partial_\nu A^\nu | k \rangle = 0$ for all states. We can therefore restrict our attention to the first term in \[8\] and use that to work out $P^\mu$. Therefore we have

$$P^\mu = -\int d^3 x \dot{A}_\rho \partial^j A^\rho$$ \quad (12)

We can now use \[3\] to convert to creation and annihilation operators. We have

$$\dot{A}_\rho = -i \int \frac{d^3 k}{\sqrt{(2\pi)^3 2\omega_k}} \sum_{r=0}^3 \left[ \epsilon_{\rho r} (k) a_r (k) e^{-ik \cdot x} - \epsilon^*_{\rho r} (k) a_r^\dagger (k) e^{ik \cdot x} \right]$$

$$\partial^j A^\rho = i \int \frac{d^3 p}{\sqrt{(2\pi)^3 2\omega_p}} p^j \sum_{s=0}^3 \left[ \epsilon_{s \rho} (p) a_s (p) e^{-ip \cdot x} - \epsilon^*_{s \rho} (p) a_s^\dagger (p) e^{ip \cdot x} \right]$$ \quad (13)

$$\dot{A}_\rho = -i \int \frac{d^3 k}{\sqrt{(2\pi)^3 2\omega_k}} \sum_{r=0}^3 \left[ \epsilon_{\rho r} (k) a_r (k) e^{-ik \cdot x} - \epsilon^*_{\rho r} (k) a_r^\dagger (k) e^{ik \cdot x} \right]$$

$$\partial^j A^\rho = i \int \frac{d^3 p}{\sqrt{(2\pi)^3 2\omega_p}} p^j \sum_{s=0}^3 \left[ \epsilon_{s \rho} (p) a_s (p) e^{-ip \cdot x} - \epsilon^*_{s \rho} (p) a_s^\dagger (p) e^{ip \cdot x} \right]$$ \quad (14)

I’ve used $k$ for momentum in the first equation and $p$ in the second (and $r$ and $s$ for the summation indexes) to avoid mixing them up when we combine them. To get the sign in these equations, remember that
When we multiply these two integrals together and integrate over $d^3x$ as in 4, we will get delta functions involving $k$ and $p$. Let’s consider the first term in 13 (which we’ll call 1) multiplied by the first term in 14 (3):

$$\langle 1 \rangle \times \langle 3 \rangle = \int \frac{d^3k}{\sqrt{2\omega_k}} \int \frac{d^3p}{\sqrt{2\omega_p}} k^j \sum_{r,s} \epsilon_{pr} (k) a^r (k) \epsilon_{ps} (p) a_s (p)$$

$$\times e^{-i(\omega_k + \omega_p)t} \frac{1}{(2\pi)^3} \int d^3x \ e^{i(k+p) \cdot x}$$

The integral in the last line is

$$\frac{1}{(2\pi)^3} \int d^3x \ e^{i(k+p) \cdot x} = \delta^3 (k + p)$$

so $k = -p, \ \omega_k = \omega_p$ and we have the result

$$\langle 1 \rangle \times \langle 3 \rangle = \frac{1}{2} \int d^3k \ k^j \sum_{r,s} \epsilon_{pr} (k) a^r (k) \epsilon_{ps} (-k) a_s (-k) e^{-2i\omega_k t}$$

Due to the $k_j$ in the integrand and the fact that $\sum_{r,s} \epsilon_{pr} (k) a^r (k) \epsilon_{ps} (-k) a_s (-k) e^{-2i\omega_k t}$ is an even function of $k$, the integral over $k_j$ gives zero. The same is true for the product of the second term in 13 (2) and the second term in 14 (4).

We are therefore left with the other two products. For $\langle 2 \rangle \times \langle 3 \rangle$ we have

$$\langle 2 \rangle \times \langle 3 \rangle = -\int \frac{d^3k}{\sqrt{2\omega_k}} \int \frac{d^3p}{\sqrt{2\omega_p}} k^j \sum_{r,s} \epsilon_{pr}^* (k) a^r_\dagger (k) \epsilon_{ps} (p) a_s (p)$$

$$\times e^{-i(\omega_k - \omega_p)t} \frac{1}{(2\pi)^3} \int d^3x \ e^{i(k-p) \cdot x}$$

Now the delta function sets $k = p$ so we have
\( (2) \times (3) = -\frac{1}{2} \int d^3k \, k^j \sum_{r,s} \epsilon_{\rho r}^* \epsilon_s^\rho a_r^+ a_s \) \hspace{1cm} (22)

where we’ve omitted the explicit dependence on \( k \) since all terms have the same dependence. We can now use the relation

\[ \epsilon_r^\mu \epsilon_s^\mu = -\zeta_r \delta_{rs} \] \hspace{1cm} (23)

to get

\[ (2) \times (3) = \frac{1}{2} \int d^3k \, k^j \sum_r \zeta_r a_r^+ a_r \] \hspace{1cm} (24)

The other product gives a similar result, except the order of the \( a^+ \) and \( a \) operators is reversed:

\[ (1) \times (4) = \frac{1}{2} \int d^3k \, k^j \sum_r \zeta_r a_r a_r^+ \] \hspace{1cm} (25)

If we impose normal ordering, we get

\[ P^j = \frac{1}{2} \int d^3k \, k^j \sum_r \zeta_r a_r a_r^+ + \frac{1}{2} \int d^3k \, k^j \sum_r \zeta_r a_r a_r^+ : \] \hspace{1cm} (26)

\[ = \int d^3k \, k^j \sum_r \zeta_r a_r a_r^+ \] \hspace{1cm} (27)

\[ = \int d^3k \, k^j \left( \sum_{r=1}^3 a_r^+ a_r - a_0^+ a_0 \right) \] \hspace{1cm} (28)

Imposing the Gupta-Bleuler condition requires that

\[ a_0 |\Psi\rangle = a_3 |\Psi\rangle \] \hspace{1cm} (29)

so two terms in the sum cancel and we’re left with

\[ P^j = \int d^3k \, k^j \sum_{r=1}^2 a_r^+ (k) a_r (k) \] \hspace{1cm} (30)

Now for \( P^0 \). From (5) we have

\[ T^{00} = \left[ -A_\rho + \left( 1 - \frac{1}{\xi} \right) g_{\rho\theta} \partial_\nu A^\nu \right] A^\rho + \frac{1}{2} \left( \partial_\mu A_\nu \right) \left( \partial^\mu A^\nu \right) - \frac{1}{2} \left( 1 - \frac{1}{\xi} \right) \left( \partial_\mu A^\mu \right)^2 \] \hspace{1cm} (31)

Invoking the condition (9) again, we can dispose of the terms involving \( \partial_\nu A^\nu \) so we’re left with
\[ T^{00} = -\dot{A}_\rho \dot{A}^\rho + \frac{1}{2} (\partial_\mu A_\nu) (\partial^\mu A^\nu) \]  
\[ = -\dot{A}_\rho \dot{A}^\rho + \frac{1}{2} \dot{A}_\rho \dot{A}^\rho + \frac{1}{2} (\partial_j A_\nu) (\partial^j A^\nu) \]  
\[ = -\frac{1}{2} \dot{A}_\rho \dot{A}^\rho - \frac{1}{2} \nabla A \cdot \nabla A \]  

(32)  

(33)  

(34)

When we now substitute using (13) and (14) we find that terms \((1) \times (3) \) and \((2) \times (4)\) contain a factor of \(\omega_k^2 - k \cdot k\) after applying the delta function. For the massless photon, we have

\[ \omega_k^2 - k \cdot k = 0 \]  

(35)

so again, these terms vanish. For \((1) \times (4) \) and \((2) \times (3)\), the delta function gives us a factor of \(\omega_k^2 + k \cdot k = 2\omega_k^2\) so we get, after following essentially the same steps as for \(P^j\) above,

\[ P^0 = \frac{1}{2} \int d^3k \omega_k \sum_r \zeta_r a_r^\dagger a_r + \frac{1}{2} \int d^3k \omega_k \sum_r \zeta_r a_r^\dagger a_r^\dagger : \]  

(36)

\[ = \int d^3k \omega_k \sum_r \zeta_r a_r^\dagger a_r \]  

(37)

\[ = \int d^3k \omega_k \left( \sum_{r=1}^3 a_r^\dagger a_r - a_0^\dagger a_0 \right) \]  

(38)

\[ = \int d^3k \omega_k \sum_{r=1}^2 a_r^\dagger (k) a_r (k) \]  

(39)

Thus in all cases, the two physical modes for a photon moving in the \(z\) direction are the two transverse modes with indexes \(r = 1, 2\).