PHOTON FIELD: COULOMB PROPAGATOR

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In their section 8.7, L&P show that the photon propagator can be written as

$$D^{\mu\nu}(k) = \frac{1}{k^2 + i\epsilon} \left[ \sum_{r=1}^{2} \epsilon^\mu_r(k) \epsilon^\nu_r(k) + D^{\mu\nu}_C(k) + D^{\mu\nu}_R(k) \right]$$  \hspace{1cm} (1)

where $D_C$ is the Coulomb part of the propagator and $D_R$ is the remainder part.

In the frame where

$$\epsilon^\mu_0 = n^\mu = (1, 0, 0, 0)$$  \hspace{1cm} (2)
$$\epsilon^\mu_3 = \frac{k^\mu - (k \cdot n) n^\mu}{\sqrt{(k \cdot n)^2 - k^2}}$$  \hspace{1cm} (3)

we can write the Fourier transform of $D^{\mu\nu}_C(k)$ as

$$D^{\mu\nu}_C(x - x') = \int \frac{d^3k}{(2\pi)^3} \frac{\epsilon^\mu_0 \epsilon^\nu_0}{k^2} e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}')} \delta(x^0 - x'^0)$$  \hspace{1cm} (4)

To do the remaining integral, we need another result. For some parameter $\lambda$, we look at the following Fourier transform.

$$\frac{e^{-r/\lambda}}{4\pi r} = \frac{1}{(2\pi)^3} \int d^3p \ e^{i\mathbf{p} \cdot \mathbf{x}} f(p)$$  \hspace{1cm} (5)

where $f(p)$ is the weighting function in the Fourier transform. We can find $f(p)$ in the usual way, by multiplying through by $e^{-iq \cdot x}$ and integrating:

$$\frac{1}{(2\pi)^3} \int d^3x \int d^3p \ e^{i(p - q) \cdot x} f(p) = \frac{1}{4\pi} \int d^3x \ e^{-i(q \cdot x) \frac{e^{-r/\lambda}}{r}}$$  \hspace{1cm} (6)

The integral over $x$ on the LHS gives us a delta function, so we have
\[
\frac{1}{(2\pi)^3} \int d^3x \int d^3p \, e^{i(p-q) \cdot x} \, f(p) = \int d^3p \, \delta^3(p-q) \, f(p)
\] 
\[= f(q) \] 
(7)

On the RHS of (6), we have \(d^3x = r^2 \sin \theta dr \, d\theta \, d\phi\). We can choose the \(z\) axis to be along \(q\) so that \(q \cdot x = qr \cos \theta\) where \(q \equiv |q|\). The integral over \(\phi\) gives \(2\pi\) so we have

\[
\frac{1}{4\pi} \int d^3x \, e^{-iq \cdot x} \frac{e^{-r/\lambda}}{r} = \frac{1}{4\pi} \int r^2 \sin \theta dr \, d\theta \, d\phi \, e^{-iqr \cos \theta} \frac{e^{-r/\lambda}}{r} 
\] 
\[= \frac{1}{2} \int_0^\infty dr \int_0^{\pi} d\theta \, e^{-iqr \cos \theta} \sin \theta \, r e^{-r/\lambda} \] 
(9)

The integral over \(\theta\) gives

\[
\int_0^{\pi} d\theta \, e^{-iqr \cos \theta} \sin \theta = \frac{1}{iqr} e^{-iqr \cos \theta} \bigg|_0^{\pi} 
\]
\[= \frac{1}{iqr} \left( e^{iqr} - e^{-iqr} \right) \] 
(12)
\[= \frac{2 \sin (qr)}{qr} \] 
(13)

Putting this into (10) we have

\[
\frac{1}{4\pi} \int d^3x \, e^{-iq \cdot x} \frac{e^{-r/\lambda}}{r} = \frac{1}{q} \int_0^\infty dr \, e^{-r/\lambda} \sin (qr) 
\] 
(14)

This integral can be looked up in tables (or, by hand, by integrating by parts twice), but I used Maple to get the result.

\[
\int_0^\infty dr \, e^{-r/\lambda} \sin (qr) = - \frac{\lambda e^{-r/\lambda} \sin (qr) + q \lambda \cos (qr))}{1 + \lambda^2 q^2} \bigg|_0^\infty 
\]
\[= \frac{q \lambda^2}{1 + \lambda^2 q^2} \] 
(15)
(16)

If we now take the limit \(\lambda \to \infty\) we have

\[
\lim_{\lambda \to \infty} \frac{q \lambda^2}{1 + \lambda^2 q^2} = \frac{1}{q} \] 
(17)

Inserting this into (14) we have
\[
\int d^3 x \frac{e^{-q \cdot x}}{4\pi r} = \lim_{\lambda \to \infty} \frac{1}{4\pi} \int d^3 x \frac{e^{-q \cdot x} e^{-\tau/\lambda}}{r}
\]  
\[= \frac{1}{|q|^2} \tag{19}\]

We can now insert this into (4). To avoid mixing up the coordinates, we’ll rewrite (19) as

\[
\int d^3 y \frac{e^{-k \cdot y}}{4\pi y} = \frac{1}{k^2} \tag{20}\]

We have

\[
\int \frac{d^3 k}{(2\pi)^3} \frac{g^{\mu 0} g^{\nu 0}}{k^2} e^{ik \cdot (x-x')} = \int \frac{d^3 k}{(2\pi)^3} g^{\mu 0} g^{\nu 0} \int \frac{d^3 y}{4\pi y} \int \frac{d^3 k}{(2\pi)^3} e^{ik \cdot (x-x')} \tag{21}\]

\[= g^{\mu 0} g^{\nu 0} \int \frac{d^3 y}{4\pi y} \int \frac{d^3 k}{(2\pi)^3} e^{ik \cdot (x-x'-y)} \tag{22}\]

The second integral is a delta function which sets \(y = x - x'\) so we end up with

\[
\int \frac{d^3 k}{(2\pi)^3} \frac{g^{\mu 0} g^{\nu 0}}{k^2} e^{ik \cdot (x-x')} = g^{\mu 0} g^{\nu 0} \int \frac{d^3 y}{4\pi y} \int \delta^3 (x-x'-y) \tag{23}\]

\[= \frac{g^{\mu 0} g^{\nu 0}}{4\pi |x-x'|} \tag{24}\]

Inserting back into (4) we have

\[
D_{\mu\nu}^{\mu\nu} (x-x') = \frac{g^{\mu 0} g^{\nu 0}}{4\pi |x-x'|} \delta (x^0 - x'^0) \tag{25}\]

The Coulomb propagator has the \(1/|x-x'|\) dependence of the classical Coulomb potential for two charges separated by the distance \(|x-x'|\).