PARITY TRANSFORMATION FOR MASSLESS FERMIONS

By requiring the Dirac Lagrangian to be invariant under parity, L&P show that the parity matrix must satisfy the three conditions

\[ P^\dagger P = 1 \] \hspace{1cm} (1)
\[ \gamma_0 P^\dagger \gamma_0 \gamma_i P = -\gamma_i \] \hspace{1cm} (2)
\[ \gamma_0 P^\dagger \gamma_0 = 1 \] \hspace{1cm} (3)

Since the last condition arises from the mass term in the Lagrangian, if the fermion is massless so that \( m = 0 \), we need not satisfy this last condition on \( P \). In this case, we can choose

\[ P = \eta_P' \gamma_0 \gamma_5 \] \hspace{1cm} (4)

where \( \eta_P' \) is a parity phase factor. We can check that this satisfies the first two conditions by using the properties of the gamma matrices. For the first condition, we have (we’ll take \( \eta_P' = \pm 1 \) in what follows):

\[ P^\dagger P = \gamma_5^\dagger \gamma_0^\dagger \gamma_0 \gamma_5 \]
\[ = \gamma_5 \gamma_0 \gamma_5 \]
\[ = \gamma_5^2 \]
\[ = 1 \] \hspace{1cm} (8)

where we’ve used the facts that \( \gamma_5^\dagger = \gamma_5 \), \( \gamma_0^\dagger = \gamma_0 \) and \( \gamma_5^2 = \gamma_0^2 = 1 \). For the second condition, we have
\[ \gamma_0 \gamma_5 \gamma_i \gamma_0 \gamma_5 \gamma_0 \gamma_5 \gamma_i = \gamma_0 \gamma_5 \gamma_i \gamma_0 \gamma_5 \gamma_i \gamma_0 \gamma_5 \gamma_i \gamma_0 \gamma_5 \gamma_i = -\gamma_0 \gamma_5 \gamma_i \gamma_0 \gamma_5 \gamma_i \gamma_0 \gamma_5 \gamma_i = -\gamma_i \gamma_0 \gamma_5 \gamma_i \gamma_0 \gamma_5 \gamma_i \gamma_0 \gamma_5 \gamma_i = -\gamma_i \gamma_0 \gamma_5 \gamma_i \gamma_0 \gamma_5 \gamma_i \gamma_0 \gamma_5 \gamma_i = -\gamma_i \]

In the third line, we used the fact that the anticommutators are \(\{\gamma_5, \gamma_i\} = \{\gamma_0, \gamma_i\} = 0\).

We can now see what effect this has on a couple of interaction terms that could appear in the Lagrangian. First, suppose we have the interaction term

\[ \mathcal{L}_{\text{int}} = -\hbar \overline{\psi} \psi \phi \]  

where \(\psi\) is a fermion field and \(\phi\) is a scalar field. The fermion field transforms according to

\[ \psi_P(x) = P \psi(\tilde{x}) \]  

where

\[ \tilde{x} \equiv (t, -x) \]

Using (4) we have

\[ \mathcal{L}_{\text{int}} \rightarrow -\hbar \overline{\psi}_P \psi P \phi P \]  

For the first factor, we have

\[ \overline{\psi}_P(x) = \psi_P^\dagger(\tilde{x}) \gamma_0 \]  

\[ = \psi^\dagger(\tilde{x}) P^\dagger \gamma_0 \]  

\[ = \overline{\psi}(\tilde{x}) \gamma_0 P^\dagger \gamma_0 \]  

\[ = \eta_P \overline{\psi}(\tilde{x}) \gamma_0 \gamma_5 \gamma_0 \gamma_5 \gamma_i \]  

\[ = \eta_P \overline{\psi}(\tilde{x}) \gamma_0 \gamma_5 \gamma_i \]

Therefore

\[ \overline{\psi} \psi \phi \rightarrow \eta_P^2 \overline{\psi}(\tilde{x}) \gamma_0 \gamma_5 \gamma_0 \gamma_5 \gamma_i \psi(\tilde{x}) \phi(\tilde{x}) \]  

\[ = -\overline{\psi}(\tilde{x}) \gamma_0 \gamma_5 \gamma_0 \gamma_5 \gamma_i \psi(\tilde{x}) \phi(\tilde{x}) \]  

\[ = -\overline{\psi}(\tilde{x}) \psi(\tilde{x}) \phi(\tilde{x}) \]

Thus for the transformation to be invariant under parity, we must have
\[ \phi_P(x) = -\phi(\bar{x}) \]  

(26)

which makes \( \phi \) a pseudoscalar field.

For the interaction term

\[ L_{\text{int}} = -h' \bar{\psi} \gamma_5 \psi \phi \]  

(27)

we have

\[
\bar{\psi} \gamma_5 \psi \phi \rightarrow \bar{\psi}(\bar{x}) \gamma_0 \gamma_5 \gamma_0 \gamma_5 \psi(\bar{x})
\]

(28)

\[
= \bar{\psi}(\bar{x}) \gamma_5 \psi(\bar{x})
\]

(29)

Thus in this case, for parity invariance we must have

\[ \phi_P(x) = +\phi(\bar{x}) \]  

(30)

making \( \phi \) a scalar field.

An interaction which combines both of the above interactions, such as

\[ L_{\text{int}} = -\bar{\psi}(h + h' \gamma_5) \psi \phi \]  

(31)

can not be parity invariant, as the two terms require opposite parities for \( \phi \).