One of the mathematical tools used in quantum field theory is the functional and its derivative, known as a functional derivative. Just as an ordinary function takes a number as input and produces a number as output, a functional takes an entire function as input and produces a number. Many functionals are defined as integrals over the input function. The notation for a functional $F$ with input function $f$ is $F[f]$. For example

$$F[f] = \int_{-1}^{1} f(x) \, dx \quad (1)$$

If $f(x) = x^2$

$$F[x^2] = \int_{-1}^{1} x^2 \, dx \quad (2)$$

$$= \frac{2}{3} \quad (3)$$

Just as a regular function has a derivative with respect to its argument, a functional can have a functional derivative with respect to its input function. In a regular derivative, the idea is to change the independent variable ($x$ for a function $f(x)$) a little bit ($dx$) and see how the function changes in response. A functional derivative changes the entire input function by a small amount $\delta f(x)$ and observes how the functional changes in response.

Obviously, there are an infinite number of ways we could change $f(x)$ in the functional; in the functional above, we might increase $f(x)$ a bit between $-1$ and $0$ and decrease it a bit between $0$ and $+1$, or we might increase or decrease it a bit over the entire range and so on. We clearly need something a bit more definite if we’re to get a consistent definition of a functional derivative.

The definition used in Lancaster & Blundell is

$$\delta f(x) = \epsilon \delta (x - x_0) \quad (4)$$
where $\delta (x - x_0)$ is the Dirac delta function and $\epsilon$ is some small number. The quantity $x_0$ is some value of $x$ within the domain of $f(x)$. The idea is that the small change in $f(x)$ occurs at one point only (at $x = x_0$). With this definition, we can now define the functional derivative as

$$\frac{\delta F}{\delta f(x_0)} (\epsilon) \equiv \lim_{\epsilon \to 0} \frac{F[f(x) + \epsilon \delta (x - x_0)] - F[f(x)]}{\epsilon}$$  \hspace{1cm} (5)$$

Note that $\delta$ is used in the notation $\frac{\delta F}{\delta f(x_0)}$ for a functional derivative, replacing $d$ in an ordinary derivative $\frac{df}{dx}$.

**Example 1.** For example, with $F[f]$ defined as in [1], we get

$$\frac{\delta F[f]}{\delta f(x_0)} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left[ \int_{-1}^{1} (f(x) + \epsilon \delta (x - x_0)) dx - \int_{-1}^{1} f(x) dx \right]$$

$$= \int_{-1}^{1} \delta (x - x_0) dx$$

The value of the derivative depends on whether $x_0$ is within the range of integration, so we get

$$\frac{\delta F[f]}{\delta f(x_0)} = \begin{cases} 1 & \text{if } -1 < x_0 < 1 \\ 0 & \text{otherwise} \end{cases}$$ \hspace{1cm} (8)$$

**Example 2.** Define the functional

$$H[f] = \int_{a}^{b} G(x, y) f(y) dy$$

Then

$$\frac{\delta H[f]}{\delta f(z)} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left[ \int_{a}^{b} G(x, y) (f(y) + \epsilon \delta (y - z)) dy - \int_{a}^{b} G(x, y) f(y) dy \right]$$

$$= \int_{a}^{b} G(x, y) \delta (y - z) dy$$

$$= G(x, z)$$ \hspace{1cm} (12)$$

assuming $a < z < b$, zero otherwise.

**Example 3.** Returning to [1], we can now find a second derivative of $F[f^3]$. We start with the first derivative:
\[ \frac{\delta F[f^3]}{\delta f(x_0)} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left[ \int_{-1}^{1} (f(x) + \epsilon \delta(x - x_0))^3 \, dx - \int_{-1}^{1} f^3(x) \, dx \right] \]

(13)

\[ = 3 \int_{-1}^{1} f^2(x) \delta(x - x_0) \, dx \]

(14)

\[ = 3f^2(x_0) \]

(15)

where in going from line 1 to line 2, we kept only the term first order in \( \epsilon \) since higher order terms vanish in the limit \( \epsilon \to 0 \). The result assumes \(-1 < x_0 < 1 \) (the answer is 0 otherwise). Now we can take a second derivative by just applying the definition again.

\[ \frac{\delta F[f^3]}{\delta f(x_0) \, \delta f(x_1)} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left[ \int_{-1}^{1} \left( \frac{\partial}{\partial y} \left( f(x) + \epsilon \delta(x - x_1) \right) \right)^2 \, dy - \int_{-1}^{1} f^2(x) \, dy \right] \]

(16)

\[ = 6f(x_0) \delta(x_0 - x_1) \]

(17)

**Example 4.** Now suppose we have the functional

\[ J[f] = \int_{a}^{b} \left( \frac{\partial f}{\partial y} \right)^2 \, dy \]

(18)

The derivative is

\[ \frac{\delta J[f]}{\delta f(x)} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left[ \int_{a}^{b} \left( \frac{\partial}{\partial y} \left( f + \epsilon \delta(y - x) \right) \right)^2 \, dy - \int_{a}^{b} \left( \frac{\partial f}{\partial y} \right)^2 \, dy \right] \]

(19)

\[ = 2 \int_{a}^{b} f'(y) \delta'(y - x) \, dy \]

(20)

where a prime indicates a derivative with respect to \( y \). We can solve this using integration by parts:

\[ \int_{a}^{b} f'(y) \delta'(y - x) \, dy = f'(y) \delta(y - x) \bigg|_{a}^{b} - \int_{a}^{b} f''(y) \delta(y - x) \, dy \]

(21)

Provided that neither \( a \) nor \( b \) coincides with \( x \), the delta function in the integrated term is zero at both limits so the first term vanishes and we’re left with

\[ \int_{a}^{b} f'(y) \delta'(y - x) \, dy = -f''(x) \]

(22)

so
\[
\frac{\delta J[f]}{\delta f(x)} = -2 \frac{\partial^2 f}{\partial x^2}
\]  \hfill (23)

if \(a < x < b\), zero otherwise.

[Incidentally, if you’re worried about switching the derivative from \(y\) to \(x\) in

\[
\int_a^b f''(y) \delta (y - x) \, dy = \int_a^b \frac{\partial^2 f}{\partial y^2} \delta (y - x) \, dy = \frac{\partial^2 f}{\partial x^2}
\]  \hfill (24)

it doesn’t matter whether we take the derivative with respect to \(y\) and then set \(y = x\) or whether we set \(y = x\) first and then take the derivative with respect to \(x\). All we’re doing is using a different variable name for the same derivative operation, so the two orders of doing things are equivalent.]

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