GENERALIZED COMMUTATOR

We’ve seen that the creation and annihilation operators obey commutation relations for bosons and anticommutation relations for fermions. We can represent both these relations by using a generalized commutator defined by

\[ [A, B]_\zeta \equiv AB - \zeta BA \tag{1} \]

for operators \( A \) and \( B \), and the parameter \( \zeta = +1 \) for bosons and \( \zeta = -1 \) for fermions. Using this notation, we have

\[ [A, B]_\zeta = [A, B] \tag{2} \]

for bosons and

\[ [A, B]_\zeta = \{A, B\} \tag{3} \]

for fermions. The commutators for fields \( \psi(x) \) are

\[ \left[ \psi(x), \psi^\dagger(y) \right]_\zeta = \delta^{(3)}(x - y) \tag{4} \]

\[ \left[ \psi(x), \psi(y) \right]_\zeta = 0 \tag{5} \]

\[ \left[ \psi^\dagger(x), \psi^\dagger(y) \right]_\zeta = 0 \tag{6} \]

Using L&B’s definition of \( V_{\text{wrong}} \) for a two-particle interaction:

\[ V_{\text{wrong}} = \frac{1}{2} \int d^3x \, d^3y \, V(x, y) \rho(y) \rho(x) \tag{7} \]

where the density operators are given by

\[ \rho(x) \rho(y) = \psi^\dagger(x) \psi(x) \psi^\dagger(y) \psi(y) \tag{8} \]

we can write \( V_{\text{wrong}} \) using the generalized commutators as follows.
\[ \rho(x) \rho(y) = \zeta \psi^\dagger(x) \psi^\dagger(y) \psi(x) \psi(y) + \delta^{(3)}(x-y) \psi^\dagger(x) \psi(y) \] (9)

\[ = \zeta^2 \psi^\dagger(x) \psi^\dagger(y) \psi(x) \psi(y) + \delta^{(3)}(x-y) \psi^\dagger(x) \psi(y) \] (10)

In the first line, we swapped middle two terms in 8 using 4, and in the last line, we swapped the last two terms \( \psi(x) \psi(y) \) using 5. Since \( \zeta^2 = 1 \) for both bosons and fermions, the final result is the same for both:

\[ \rho(x) \rho(y) = \psi^\dagger(x) \psi^\dagger(y) \psi(x) \psi(y) + \delta^{(3)}(x-y) \psi^\dagger(x) \psi(y) \] (11)

This contains an extra self-energy term (the term containing the delta function), which can be avoided by using normal ordering.