MAXWELL’S EQUATIONS FROM THE ELECTROMAGNETIC FIELD TENSOR

We’ve seen before how Maxwell’s equations can be obtained from the electromagnetic field tensor

$$F_{\mu \nu} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{bmatrix}$$

(1)

Just to review this, see this post where we show that the relation

$$\partial_\lambda F_{\mu \nu} + \partial_\mu F_{\nu \lambda} + \partial_\nu F_{\lambda \mu} = 0$$

(2)

yields two of Maxwell’s equations:

$$\nabla \cdot B = 0$$

(3)

$$\nabla \times E + \frac{\partial B}{\partial t} = 0$$

(4)

The other relation is

$$\partial_\mu F^{\mu \nu} = J^\nu$$

(5)

gives the other two Maxwell equations

$$\nabla \cdot E = \rho$$

(6)

$$\nabla \times B - \frac{\partial E}{\partial t} = J$$

(7)

From the facts that second derivatives are symmetric and $F_{\mu \nu}$ is antisymmetric, we have
\[ \partial_\mu \partial_\nu F^{\mu\nu} = \partial_\nu \partial_\mu F^{\mu\nu} \] (8)
\[ = -\partial_\nu \partial_\mu F^{\nu\mu} \] (9)
\[ = -\partial_\mu \partial_\nu F^{\mu\nu} \] (10)

In the last line, we merely swapped the indices \( \mu \) and \( \nu \) which is permissible since they are just dummy indices that are summed. Thus \( \partial_\mu \partial_\nu F^{\mu\nu} \) is equal to its negative, so it must be zero:
\[ \partial_\mu \partial_\nu F^{\mu\nu} = 0 \] (11)

Thus from (5) we have
\[ \partial_\nu J^\nu = 0 \] (12)

Since \( J^\nu \) is the four-vector
\[ J^\nu = [\rho, \mathbf{J}] \] (13)

and
\[ \partial_\nu = \left[ \frac{\partial}{\partial t}, -\nabla \right] \] (14)
this gives
\[ \frac{\partial \rho}{\partial t} = \nabla \cdot \mathbf{J} \] (15)

which is the usual continuity equation for charge. The rate of change of charge density at a given point is equal to the divergence of the current, which carries charge into or away from that point.

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