**TIME EVOLUTION OPERATOR**

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In the Schrödinger picture (where the time dependence of a quantum system resides in the wave functions rather than the operators), we can write the time evolution of a wave function as

\[
\psi(t_2) = U(t_2, t_1) \psi(t_1)
\]

where the time evolution operator \( U(t_2, t_1) \) operates on a wave function \( \psi(t_1) \) at time \( t_1 \) and 'evolves' it to the wave function \( \psi(t_2) \) at time \( t_2 \).

L&B list five properties that a time evolution operator should have. One of these properties is

\[
i \frac{d}{dt_2} U(t_2, t_1) = HU(t_2, t_1)
\]

where \( H \) is the Hamiltonian operator. This can be formally integrated to give

\[
U(t_2, t_1) = e^{-iH(t_2-t_1)}
\]

We can now demonstrate that this form does satisfy the five required properties.

**Property 1.** \( U(t_1, t_1) = 1 \). From (3) with \( t_2 = t_1 \) we have

\[
U(t_1, t_1) = e^{-iH(t_1-t_1)} = e^0 = 1
\]

**Property 2.** \( U(t_3, t_2)U(t_2, t_1) = U(t_3, t_1) \) (the composition law). We have

\[
U(t_3, t_2)U(t_2, t_1) = e^{-iH(t_3-t_2)}e^{-iH(t_2-t_1)}
\]

\[
= e^{-iH(t_3-t_1)}
\]

\[
= U(t_3, t_1)
\]

**Property 3.** This is just the differential equation (2) from which (3) was derived, so it's automatically satisfied.

**Property 4.** \( U(t_1, t_2) = U^{-1}(t_2, t_1) \) (the inverse property). We have

L&B denote an operator by putting a hat over it, as in \( \hat{U}(t_2, t_1) \), but to save typing I'll omit the hat since it's fairly obvious from the context what the operators are.
\begin{align}
U(t_1, t_2) U(t_2, t_1) &= e^{-iH(t_1-t_2)} e^{-iH(t_2-t_1)} \\
&= 1
\end{align}

Therefore \( U(t_1, t_2) = U^{-1}(t_2, t_1) \). Taking the inverse of a time evolution operator evolves the state backwards in time.

**Property 5.** \( U^\dagger(t_2, t_1) U(t_2, t_1) = 1 \) (the unitary property). In other words, the hermitian conjugate of \( U \) is also its inverse. We have

\begin{align}
U^\dagger(t_2, t_1) &= e^{iH^\dagger(t_2-t_1)} \\
&= e^{iH(t_2-t_1)} \\
&= U^{-1}(t_2, t_1)
\end{align}

The second line follows because the Hamiltonian is hermitian, so \( H^\dagger = H \). Thus all five properties are satisfied.