NOETHER CURRENT FOR SYSTEM OF MANY SCALAR FIELDS

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L&B derive the Noether current $J^\mu$ for a Lagrangian that depends on a single scalar field $\phi(x)$ in their Chapter 10. Problem 10.2 asks us to go through the same derivation for a Lagrangian that depends on $N$ scalar fields $\phi_1, \ldots, \phi_N$ so that

$$L = L(\phi_1, \ldots, \phi_N; \partial_\mu \phi_1, \ldots, \partial_\mu \phi_N; x^\mu)$$  \hfill (1)

The derivation is very similar to that given in the book. The main difference is that we need to use the chain rule for derivatives in a few places. We’ll run through the derivation here. We get (where repeated indices are summed, with $a$ being summed from 1 to $N$):

$$\delta L = \frac{\partial L}{\partial \phi^a} \delta \phi^a + \frac{\partial L}{\partial (\partial_\mu \phi^a)} \delta (\partial_\mu \phi^a)$$ \hfill (2)

$$= \frac{\partial L}{\partial \phi^a} \delta \phi^a + \Pi_\mu^a \delta (\partial_\mu \phi^a)$$ \hfill (3)

where

$$\Pi_\mu^a \equiv \frac{\partial L}{\partial (\partial_\mu \phi^a)}$$ \hfill (4)

We now use the product rule to write

$$\partial_\mu (\Pi_\mu^a \delta \phi^a) = \Pi_\mu^a \delta (\partial_\mu \phi^a) + \partial_\mu (\Pi_\mu^a) \delta \phi^a$$ \hfill (5)

Using this, we rewrite as

$$\delta L = \frac{\partial L}{\partial \phi^a} \delta \phi^a + \partial_\mu (\Pi_\mu^a \delta \phi^a) - \partial_\mu (\Pi_\mu^a) \delta \phi^a$$ \hfill (6)

$$= \left( \frac{\partial L}{\partial \phi^a} - \partial_\mu (\Pi_\mu^a) \right) \delta \phi^a + \partial_\mu (\Pi_\mu^a \delta \phi^a)$$ \hfill (7)
We now require the system to satisfy the Euler-Lagrange equations of motion, which state
\[
\frac{\partial \mathcal{L}}{\partial \dot{\phi}^a} - \partial_\mu (\Pi^\mu_a) = 0
\] (8)
so that we have
\[
\delta \mathcal{L} = \partial_\mu (\Pi^\mu_a \delta \phi^a)
\] (9)

We now consider a translation along a spacetime vector $b^\mu$ by an amount $\lambda b^\mu$, where $\lambda$ is some parameter that can be varied to vary the size of the translation. Under this transformation the $N$ fields will transform as
\[
\phi_a (x^\mu) \rightarrow \phi_a (x^\mu + \lambda b^\mu)
\] (10)

We then define the quantity
\[
D\phi^a \equiv \left. \frac{\partial \phi^a}{\partial \lambda} \right|_{\lambda=0} = b^\mu \partial_\mu \phi^a
\] (11)
so that
\[
\delta \phi^a = D\phi^a \delta \lambda
\] (12)

The goal is that the action
\[
S = \int d^4 x \, \mathcal{L}
\] (13)
remain stationary under the symmetry transformation (since we’re assuming that the system is invariant under this symmetry transformation). One way of ensuring this is to require $\delta \mathcal{L} = 0$, but since the change in action is
\[
\delta S = \int d^4 x \, \delta \mathcal{L}
\] (14)
where the integral extends over all spacetime. We can allow $\delta \mathcal{L}$ to be some total divergence, provided that we can apply the divergence theorem to convert the spacetime integral to a surface integral, which as usual vanishes as we go to infinity. That is, we can let
\[
\delta \mathcal{L} = (\partial_\mu W^\mu) \delta \lambda
\] (15)
for some function $W^\mu (x)$. For a given $W^\mu$, the effect this has on the action integral will be a change in the total action of some constant value (which could be zero), so if $\delta S = 0$ when $\delta \mathcal{L} = 0$ it will still be zero if we add on the term $(\partial_\mu W^\mu) \delta \lambda$ since we’re not varying any of the fields in doing this.

Equating this with we get the condition
\[ \partial_\mu (\Pi_\mu^a \delta \phi^a - W^\mu \delta \lambda) = \partial_\mu (\Pi_\mu^a D\phi^a - W^\mu) \delta \lambda \]
\[ = 0 \]  \hspace{1cm} (16)  \hspace{1cm} (17)

which gives the Noether current

\[ J^\mu_N (x) = \Pi_\mu^a D\phi^a - W^\mu \]  \hspace{1cm} (18)

which satisfies the conservation rule

\[ \partial_\mu J^\mu_N (x) = 0 \]  \hspace{1cm} (19)