COMMUTATORS OF COMPLEX SCALAR FIELDS AT GENERAL TIMES

The complex scalar field \( \psi(x) \) has the mode expansion

\[
\psi(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2E_p}} \left( a(p) e^{-ip \cdot x} + b^\dagger(p) e^{ip \cdot x} \right)
\]

\[
\psi^\dagger(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2E_p}} \left( b(p) e^{-ip \cdot x} + a^\dagger(p) e^{ip \cdot x} \right)
\]

We can work out the commutator of the field at different spacetime points using the commutators for the operators

\[
[a(p), a^\dagger(q)] = [b(p), b^\dagger(q)] = \delta^{(3)}(p-q)
\]

with all other commutators being zero. To simplify the notation, I’ll use

\[
a_p \equiv a(p)
\]

and similar for other operators.

We get

\[
\left[ \psi(x), \psi^\dagger(y) \right] = \frac{1}{2(2\pi)^3} \int \frac{d^3p d^3q}{\sqrt{E_p E_q}} \left\{ a_p a_q^\dagger e^{-i(p-x-q)y} - b_q b_p^\dagger e^{i(p-x-q)y} - \delta^{(3)}(p-q) e^{-i(p-x-q)y} - \delta^{(3)}(p-q) e^{i(p-x-q)y} \right\}
\]

\[
= \frac{1}{2(2\pi)^3} \int \frac{d^3p d^3q}{\sqrt{E_p E_q}} \left\{ -\delta^{(3)}(p-q) e^{-i(p-x-q)y} - \delta^{(3)}(p-q) e^{i(p-x-q)y} \right\}
\]

\[
= \frac{1}{2(2\pi)^3} \int \frac{d^3p}{E_p} \left\{ e^{-ip \cdot (x-y)} - e^{ip \cdot (x-y)} \right\}
\]

Note that if \( x^0 = y^0 \) (that is, the times are equal), then
and we get

\[
\left[ \psi(t, x), \psi^\dagger(t, y) \right] = \frac{1}{2(2\pi)^3} \int \frac{d^3p}{E_p} \left\{ e^{-ip(x-y)} - e^{ip(x-y)} \right\}
\]

(9)

Since we’re integrating over all \(p\), we can swap \(p \to -p\) in the second term with the result

\[
\left[ \psi(t, x), \psi^\dagger(t, y) \right] = 0
\]

(10)

This is the complex scalar field analog to the result we got earlier for a real scalar field, which shows that for two spacetime points with a spacelike separation, the commutator of the fields at these two points is zero, which is required to preserve causality.

In their section 12.3, L&B consider the non-relativistic limit of a complex scalar field. They define the field as

\[
\phi(x, t) = \Psi(x, t) e^{-imc^2t/\hbar}
\]

(11)

where the rest energy has been factored out, since in the non-relativistic case, the rest energy is by far the dominant part of the energy. They then show that we can write the other part of the field as

\[
\Psi(x) = \int \frac{d^3p}{(2\pi)^{3/2}} \alpha_p e^{-ip \cdot x}
\]

(12)

Using this form, we have for the commutator at general spacetime points:

\[
\left[ \Psi(x), \Psi^\dagger(y) \right] = \frac{1}{(2\pi)^3} \int d^3p \int d^3q \left[ \alpha_p, \alpha_q^\dagger \right] e^{-i(p \cdot x - q \cdot y)}
\]

(13)

\[
= \frac{1}{(2\pi)^3} \int d^3p \int d^3q \delta^{(3)}(p - q) e^{-i(p \cdot x - q \cdot y)}
\]

(14)

\[
= \frac{1}{(2\pi)^3} \int d^3p e^{-ip \cdot (x-y)}
\]

(15)

At equal times, we have

\[
\left[ \Psi(t, x), \Psi^\dagger(t, y) \right] = \frac{1}{(2\pi)^3} \int d^3p e^{-ip \cdot (x-y)}
\]

(16)

\[
= \delta^{(3)}(x - y)
\]

(17)
which agrees with L&B’s equation 12.27.

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