NONRELATIVISTIC COMPLEX SCALAR FIELD:
COMMUTATOR OF NUMBER WITH PHASE

In their section 12.3, L&B write the nonrelativistic version of the complex scalar field in the form

$$\Psi(x) = \sqrt{\rho(x)} e^{i\theta(x)} \quad (1)$$

so that $\rho(x) = |\Psi(x)|^2$ is the probability density and $\theta(x)$ is a spacetime-dependent phase. Both $\rho$ and $\theta$ are taken to be real functions.

Using this form, the nonrelativistic Lagrangian becomes

$$\mathcal{L} = \frac{i}{2} \partial_0 \rho - \rho \partial_0 \theta - \frac{1}{2m} \left[ \frac{1}{4\rho} (\nabla \rho)^2 + \rho (\nabla \theta)^2 \right] - \frac{g}{2} \rho^2 \quad (2)$$

where $g$ represents the interaction term.

If we make the global transformation $\theta \to \theta + \alpha$ where $\alpha$ is a constant, the Lagrangian is unchanged, since only the derivatives of $\theta$ enter into the Lagrangian.

To apply Noether’s theorem to this transformation, we have

$$D\theta = \left. \frac{\partial \theta'}{\partial \alpha} \right|_{\alpha=0} \quad (3)$$

$$= \left. \frac{\partial (\theta + \alpha)}{\partial \alpha} \right|_{\alpha=0} \quad (4)$$

$$= 1 \quad (5)$$

Since we’re not transforming $\rho$, $D\rho = 0$. The zero component of the Noether current is then
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\[ J^0_N = \Pi_0^0 D\theta + \Pi_\rho^0 D\rho \]  
\[ = \frac{\partial L}{\partial (\partial_0 \theta)} D\theta + 0 \]  
\[ = -\rho(x) \times 1 \]  
\[ = -\rho(x) \]  

which gives a conserved charge of

\[ Q_N = \int d^3x \ J^0_N(x) \]  
\[ = -\int d^3x \ \rho(x) \]  

This gives what L&B call the 'conventional' charge

\[ Q_{Nc} = -:Q_N: \]  
\[ =:\int d^3x \ \rho(x): \]  

We can now use the commutator of conserved charge together with 5 to write

\[ [Q_N, \theta] = -i D\theta = -i \]  

If we define

\[ N(t) = \int d^3x \ \rho(x) \]  

to be the total number of particles, then

\[ N(t) = -Q_N \]  

and

\[ [Q_N, \theta] = -[N(t), \theta(x,t)] = -i \]  

or

\[ [N(t), \theta(x,t)] = i \]