THREE-COMPONENT FIELD: NUMBER OPERATOR

In their Chapter 13, L&B consider a field $\Phi$ which has 3 components that could represent 3 different states of a particle. They give the Lagrangian as

$$\mathcal{L} = \frac{1}{2} (\partial^\mu \Phi) \cdot (\partial_\mu \Phi) - \frac{m^2}{2} \Phi \cdot \Phi$$  \hspace{1cm} (1)

This Lagrangian is invariant under 3-d rotations which they show leads to a conserved charge vector $Q_{Nc}$:

$$Q_{Nc} = \int d^3 x \ (\Phi \times \partial_0 \Phi)$$  \hspace{1cm} (2)

where the integrand is the standard cross product of two 3-d vectors.

The field vector $\Phi$ is assumed to have the mode expansion for each component given by

$$\Phi_\alpha = \int \frac{d^3 p}{(2\pi)^{3/2} \sqrt{2E_p}} \left( a_{p\alpha} e^{-ip \cdot x} + a^\dagger_{p\alpha} e^{ip \cdot x} \right)$$  \hspace{1cm} (3)

where the index $\alpha = 1, 2, 3$ specifies which of the 3 states of the particle we’re looking at.

With this mode decomposition, the conserved charge then has components

$$Q^\alpha_{Nc} = -i \int \frac{d^3 p}{2E_p} \varepsilon^{abc} a^\dagger_{pb} a_{pc}$$  \hspace{1cm} (4)

For example

$$Q^3_{Nc} = -i \int \frac{d^3 p}{2E_p} \left( a^\dagger_{p1} a_{p2} - a^\dagger_{p2} a_{p1} \right)$$  \hspace{1cm} (5)

The first part of the problem is to write the conserved charge in the compact form

$$Q_{Nc} = \int d^3 p \ A^\dagger_p J_A p$$  \hspace{1cm} (6)
where \( \mathbf{A} = (a_{p1}, a_{p2}, a_{p3}) \) is the vector of annihilation operators and \( \mathbf{J} \) is a 3-component vector where each component is a 3 \( \times \) 3 matrix consisting of the spin-1 angular momentum matrices given in L&B’s Chapter 9. Actually, the matrices they give there are 4 \( \times \) 4, so I’m assuming they want us to use the 3 \( \times \) 3 sub-matrix containing the spatial components. These matrices are

\[
J_x = -i \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & -1 \\
0 & 1 & 0
\end{bmatrix} \tag{7}
\]

\[
J_y = -i \begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
-1 & 0 & 0
\end{bmatrix} \tag{8}
\]

\[
J_z = -i \begin{bmatrix}
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} \tag{9}
\]

We can now work out the matrix term \( \mathbf{A}^{\dagger} \mathbf{J} \mathbf{A} \). (I’ll drop the \( \mathbf{p} \) subscript to avoid clutter.) First, we have

\[
\mathbf{J} \mathbf{A} = -i \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & -a_1 \\
0 & a_1 & 0
\end{bmatrix} - i \begin{bmatrix}
0 & 0 & a_2 \\
0 & 0 & 0 \\
-a_2 & 0 & 0
\end{bmatrix} - i \begin{bmatrix}
0 & -a_3 & 0 \\
-a_3 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

\[
= -i \begin{bmatrix}
0 & -a_3 & a_2 \\
a_3 & 0 & -a_1 \\
-a_2 & a_1 & 0
\end{bmatrix} \tag{10}
\]

Next, we have

\[
\mathbf{A}^{\dagger} \mathbf{J} \mathbf{A} = -i \begin{bmatrix}
a_1^\dagger & a_2^\dagger & a_3^\dagger
\end{bmatrix} \begin{bmatrix}
0 & -a_3 & a_2 \\
a_3 & 0 & -a_1 \\
-a_2 & a_1 & 0
\end{bmatrix}
\]

\[
= -i \begin{bmatrix}
a_1^\dagger a_3 - a_2^\dagger a_2 \\
a_3^\dagger a_1 - a_1^\dagger a_3 \\
-a_2^\dagger a_1 - a_1^\dagger a_2
\end{bmatrix} \tag{12}
\]

By comparing this with \([4]\) we see that the 3 components of \( \mathbf{Q}_{N_c} \) match up properly.

Next, we want to change to a different set of creation and annihilation operators, namely those given in Exercise 3.3.
\[ \hat{b}^+_1 \equiv -\frac{1}{\sqrt{2}} \left( \hat{a}^+_1 + i\hat{a}^+_2 \right) \]  
\[ \hat{b}^+_0 \equiv \hat{a}^+_3 \]  
\[ \hat{b}^+_{-1} \equiv \frac{1}{\sqrt{2}} \left( \hat{a}^+_1 - i\hat{a}^+_2 \right) \]  
\[ \hat{b}_1 \equiv -\frac{1}{\sqrt{2}} (\hat{a}_1 - i\hat{a}_2) \]  
\[ \hat{b}_0 \equiv \hat{a}_3 \]  
\[ \hat{b}_{-1} \equiv \frac{1}{\sqrt{2}} (\hat{a}_1 + i\hat{a}_2) \]  

Using \( B = (b_1, b_0, b_{-1}) \) we are to find the new set of matrices \( J \) so that

\[ Q_{Nc} = \int d^3p \, B_p^t JB_p \]  

There may be some quick way of doing this, but it seems that we need to express the \( a_i \) operators in terms of \( b_j \) and proceed from there. We have

\[ a_1 = \frac{1}{\sqrt{2}} (b_{-1} - b_1) \]  
\[ a_2 = -\frac{i}{\sqrt{2}} (b_{-1} + b_1) \]  
\[ a_3 = b_0 \]  
\[ a_1^+ = \frac{1}{\sqrt{2}} (b_{-1}^+ - b_1^+) \]  
\[ a_2^+ = \frac{i}{\sqrt{2}} (b_{-1}^+ + b_1^+) \]  
\[ a_3^+ = b_0^+ \]  

With these terms, we have

\[ a_2^+ a_3 - a_3^+ a_2 = -\frac{i}{\sqrt{2}} \left( b_{-1}^+ b_0 - b_0^+ (b_1 + b_{-1}) + b_{-1}^+ b_0 \right) \]  
\[ a_3^+ a_1 - a_1^+ a_3 = \frac{1}{\sqrt{2}} \left( b_{-1}^+ b_0 - b_0^+ (b_1 - b_{-1}) - b_{-1}^+ b_0 \right) \]  
\[ a_1^+ a_2 - a_2^+ a_1 = i \left( b_{-1}^+ b_1 - b_{-1}^+ b_{-1} \right) \]

Pulling out the creation operators we have
\[
B_p^\dagger JB_p = -i \begin{bmatrix}
    b_1^\dagger & b_0^\dagger & b_{-1}^\dagger
\end{bmatrix}
\begin{bmatrix}
    -\frac{i}{\sqrt{2}}b_0 & \frac{1}{\sqrt{2}}b_0 & ib_1 \\
    \frac{i}{\sqrt{2}}(b_1 + b_{-1}) & -\frac{1}{\sqrt{2}}(b_1 - b_{-1}) & 0 \\
    -\frac{i}{\sqrt{2}}b_0 & -\frac{1}{\sqrt{2}}b_0 & -ib_{-1}
\end{bmatrix}
\] (30)

We now need to express the $3 \times 3$ matrix on the RHS as a product of 3 separate matrices multiplied into $B = (b_1, b_0, b_{-1})$. We therefore have

\[
J_1 = -i \begin{bmatrix}
    0 & 0 & i \\
    \frac{i}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\
    0 & 0 & 0
\end{bmatrix} = \begin{bmatrix}
    0 & 0 & 1 \\
    \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\
    0 & 0 & 0
\end{bmatrix}
\] (31)

\[
J_2 = -i \begin{bmatrix}
    -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\
    0 & 0 & 0 \\
    -\frac{i}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0
\end{bmatrix} = \begin{bmatrix}
    -\frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 \\
    0 & 0 & 0 \\
    -\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0
\end{bmatrix}
\] (32)

\[
J_3 = -i \begin{bmatrix}
    0 & 0 & 0 \\
    \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\
    0 & 0 & -i
\end{bmatrix} = \begin{bmatrix}
    0 & 0 & 0 \\
    \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 \\
    0 & 0 & -1
\end{bmatrix}
\] (33)

Pingbacks

Pingback: Massive electromagnetism: Polarization vectors