ELECTROMAGNETISM: LAGRANGIAN USING PROJECTION OPERATORS

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In their section 13.3, L&B show that the projection of a four-vector along a given four-momentum $p^\mu$ is given by the longitudinal projection operator

$$P_{L}^{\mu \nu} = \frac{p^\mu p^\nu}{p^2}$$  \hspace{1cm} (1)

The projection transverse to the momentum is then given by

$$P_{T}^{\mu \nu} = g^{\mu \nu} - \frac{p^\mu p^\nu}{p^2}$$  \hspace{1cm} (2)

The Lagrangian density for a free electromagnetic field is

$$\mathcal{L} = -\frac{1}{4} F^{\mu \nu} F_{\mu \nu}$$  \hspace{1cm} (3)

$$= -\frac{1}{4} \left( \partial^\mu A^\nu - \partial^\nu A^\mu \right) \left( \partial_\mu A_\nu - \partial_\nu A_\mu \right)$$  \hspace{1cm} (4)

$$= -\frac{1}{4} \left( \partial^\mu A^\nu \partial_\mu A_\nu + \partial^\nu A^\mu \partial_\nu A_\mu - \partial^\nu A^\mu \partial_\mu A_\nu - \partial^\mu A^\nu \partial_\nu A_\mu \right)$$  \hspace{1cm} (5)

$$= -\frac{1}{4} \left( 2 \partial^\mu A^\nu \partial_\mu A_\nu - 2 \partial^\nu A^\mu \partial_\nu A_\mu \right)$$  \hspace{1cm} (6)

$$= -\frac{1}{2} \left( \partial^\mu A^\nu \partial_\mu A_\nu - \partial^\nu A^\mu \partial_\nu A_\mu \right)$$  \hspace{1cm} (7)

To get the fourth line, we swapped $\mu \leftrightarrow \nu$ in the second and fourth terms in the third line, which is permissible since these indexes are summed.

To express $\mathcal{L}$ in the form given by L&B’s equation 13.43, we need to use the old trick of adding in a total divergence to the Lagrangian. This is allowed since we’re assuming that the total Lagrangian is the integral of $\mathcal{L}$ over all space and, using Gauss’s theorem, the integral of a total divergence converts to a surface integral which goes to zero at infinity. In this case, we consider the term
\[ K_\mu = \frac{1}{2} A^\nu \left( \partial_\mu A_\nu - \partial_\nu A_\mu \right) \] (8)

The divergence gives us

\[ \partial^\mu K_\mu = \frac{1}{2} \partial^\mu A^\nu \left( \partial_\mu A_\nu - \partial_\nu A_\mu \right) + \frac{1}{2} A^\nu \left( \partial^\mu \partial_\mu A_\nu - \partial_\nu \partial^\mu A_\mu \right) \] (9)

We now observe that the first term in (9) is the negative of \( \mathcal{L} \) as given in (7), so we can write

\[ \mathcal{L} = \frac{1}{2} A^\nu \left( \partial^\mu \partial_\mu A_\nu - \partial_\nu \partial^\mu A_\mu \right) - \partial^\mu K_\mu \] (10)

and since we can throw away any total divergence that appears in the Lagrangian density, we can simplify this to

\[ \mathcal{L} = \frac{1}{2} A^\nu \left( \partial^\mu \partial_\mu A_\nu - \partial_\nu \partial^\mu A_\mu \right) \] (11)

\[ = \frac{1}{2} A^\nu \left( \partial^2 A_\nu - \partial_\nu \partial^\mu A_\mu \right) \] (12)

To relate this to the projection operator \( P_T \) in (2), we need to note that \( \mathcal{L} \) is given as a function of \( x \) (since all the fields \( A^\mu \) are functions of \( x \)), not \( p \), so we need to write \( P_T \) in spacetime coordinates, instead of momentum coordinates. The spacetime representation of \( p^\mu \) is \( i\partial^\mu \), so in these coordinates, we have

\[ P_T^{\mu\nu} = g^{\mu\nu} - \frac{(i\partial^\mu)(i\partial^\nu)}{(i\partial)^2} \] (13)

\[ = g^{\mu\nu} - \frac{\partial^\mu \partial^\nu}{\partial^2} \] (14)

The term containing the differential operators is assumed to act on whatever field is written to the right of \( P_T \).

We can now write (12) as
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\[ \mathcal{L} = \frac{1}{2} A^\nu \left( \partial_\nu A_\nu - \partial_\nu \partial_\mu A^\mu \right) \] (15)

\[ \begin{align*}
&= \frac{1}{2} A_\nu \partial_\nu^2 A^\nu - \frac{1}{2} A^\nu \partial_\nu \partial_\mu A^\mu \\
&= \frac{1}{2} A^\mu g_{\mu \nu} \partial_\nu^2 A^\nu - \frac{1}{2} A^\mu \partial_\mu \partial_\nu A^\nu \\
&= \frac{1}{2} A^\mu \left( g_{\mu \nu} - \frac{\partial_\mu \partial_\nu}{\partial^2} \right) \partial^2 A^\nu \\
&= \frac{1}{2} A^\mu P^T_{\mu \nu} \partial^2 A^\nu
\end{align*} \] (16) (17) (18) (19)

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