In problem 14.2, L&B state that it can be shown using Noether’s theorem that the operator $S^z$, whose eigenvalue is the $z$ component of the photon’s spin, obeys the commutation relation

$$\left[ S^z, a^\dagger_{q\lambda} \right] = i\epsilon^{\mu=1*}_\lambda(q) a^\dagger_{q\lambda=2} - i\epsilon^{\mu=2*}_\lambda(q) a^\dagger_{q\lambda=1}$$ (1)

where $\epsilon^{\mu}_\lambda(q)$ is the polarization vector where we’re taking the two polarization vectors to be transverse:

$$\epsilon^{\mu}_{\lambda=1} = (0, 1, 0, 0) \quad (2)$$
$$\epsilon^{\mu}_{\lambda=2} = (0, 0, 1, 0) \quad (3)$$

The creation operator $a^\dagger_{q\lambda}$ creates a photon of momentum $q$ in polarization state $\lambda$.

We can define creation operators for left L and right R circular polarization states by

$$b^\dagger_{qR} = -\frac{1}{\sqrt{2}} \left( a^\dagger_{q1} + ia^\dagger_{q2} \right) \quad (4)$$
$$b^\dagger_{qL} = \frac{1}{\sqrt{2}} \left( a^\dagger_{q1} - ia^\dagger_{q2} \right) \quad (5)$$

Using (1) we can work out the commutators of $S^z$ with these creation operators. We have
\[ \left[ S^z, b_{qR}^\dagger \right] = -\frac{1}{\sqrt{2}} \left( \left[ S^z, a_{q1}^\dagger \right] + i \left[ S^z, a_{q2}^\dagger \right] \right) \]
\[ = -\frac{1}{\sqrt{2}} \left( i\epsilon_1^{*} a_{q2}^\dagger - i\epsilon_1^{*} a_{q1}^\dagger + i \left( i\epsilon_2^{*} a_{q2}^\dagger - i\epsilon_2^{*} a_{q1}^\dagger \right) \right) \]
\[ = -\frac{1}{\sqrt{2}} \left( a_{q1}^\dagger (i\epsilon_1^{*} + 2\epsilon_2^{*}) + a_{q2}^\dagger (i\epsilon_1^{*} - \epsilon_2^{*}) \right) \]
\[ = -\frac{1}{\sqrt{2}} \left( a_{q1}^\dagger + ia_{q2}^\dagger \right) \]
\[ = b_{qR}^\dagger \] (10)

where to get the fourth line, we inserted the components of the \( \epsilon \) vectors using (2).

Similarly, we can get the other commutator:

\[ \left[ S^z, b_{qL}^\dagger \right] = \frac{1}{\sqrt{2}} \left( a_{q1}^\dagger (-i\epsilon_2^{*} + 2\epsilon_1^{*}) + a_{q2}^\dagger (i\epsilon_1^{*} + \epsilon_2^{*}) \right) \]
\[ = \frac{1}{\sqrt{2}} \left( -a_{q1}^\dagger + ia_{q2}^\dagger \right) \]
\[ = -b_{qL}^\dagger \] (13)

Using these commutators, we can work out the effect of \( S^z \) on a state containing a single circularly polarized photon. Such a state with a right-polarized photon is given by \( b_{qR}^\dagger |0\rangle \) so we have, using (10)

\[ S^z b_{qR}^\dagger |0\rangle = \left( b_{qR}^\dagger + b_{qR}^\dagger S^z \right) |0\rangle \]
\[ = b_{qR}^\dagger |0\rangle + 0 \]
\[ = b_{qR}^\dagger |0\rangle \]

where in the second line, we used the fact that the spin operator gives zero when acting on the vacuum state.

Similarly for \( b_{qL}^\dagger \) we have, using (13)

\[ S^z b_{qL}^\dagger |0\rangle = \left( -b_{qL}^\dagger + b_{qL}^\dagger S^z \right) |0\rangle \]
\[ = -b_{qL}^\dagger |0\rangle + 0 \]
\[ = -b_{qL}^\dagger |0\rangle \]

Thus the \( z \) component of spin has eigenvalues \( \pm 1 \).