GREEN FUNCTION FOR A FORCED HARMONIC OSCILLATOR

We've looked at using a Green function to solve the forced harmonic oscillator before, but in that case the forcing function was a delta function. Here we'll look at solving the problem a bit more generally.

Suppose we have a forced harmonic oscillator whose amplitude $A$ obeys the differential equation

$$m \frac{\partial^2}{\partial t^2} A(t - u) + m\omega_0^2 A(t - u) = \tilde{F}(\omega) e^{-i\omega(t-u)}$$

(1)

We can show by direct substitution that the solution is

$$A(t - u) = -\frac{\tilde{F}(\omega) e^{-i\omega(t-u)}}{m} \frac{1}{\omega^2 - \omega_0^2} + B(t)$$

(2)

where $B(t)$ is a solution to the homogeneous equation, that is:

$$m\ddot{B}(t) + m\omega_0^2 B(t) = 0$$

(3)

We have

$$\frac{\partial^2}{\partial t^2} A(t - u) = -\frac{\tilde{F}(\omega) e^{-i\omega(t-u)}}{m} \frac{1}{\omega^2 - \omega_0^2} \omega^2 + \dot{B}(t)$$

(4)

Substituting these two results into (1) we have

$$m \frac{\partial^2}{\partial t^2} A(t - u) + m\omega_0^2 A(t - u) = -\frac{\tilde{F}(\omega) e^{-i\omega(t-u)}}{m} \frac{1}{\omega^2 - \omega_0^2} (\omega^2 - \omega_0^2) + m\dot{B} + m\omega_0^2 B$$

(5)

$$= \tilde{F}(\omega) e^{-i\omega(t-u)} + 0$$

(6)

The Green function for (1) is given by

$$m \left( \partial^2_t + \omega_0^2 \right) G(t, u) = \delta(t - u)$$

(7)

We can write the delta function as an integral in the usual way to get
\[ m (\partial_t^2 + \omega_0^2) G(t, u) = \frac{1}{2\pi} \int d\omega e^{-i\omega(t-u)} \] (8)

We therefore have

\[ m (\partial_t^2 + \omega_0^2) \left[ \frac{1}{2\pi} \int d\omega e^{-i\omega(t-u)} \right] = \frac{m}{2\pi} \int d\omega \left( \omega_0^2 - \omega^2 \right) e^{-i\omega(t-u)} \] (9)

Therefore if we take

\[ G(t, u) = -\frac{1}{2\pi m} \int d\omega \frac{e^{-i\omega(t-u)}}{\omega^2 - \omega_0^2} \] (10)

this satisfies (7). To get the general solution for the Green function, we can add on \( B(t) \), since because of (3) this will add zero to the LHS of (7). Thus the general Green function is given by

\[ G(t, u) = -\frac{1}{2\pi m} \int d\omega \frac{e^{-i\omega(t-u)}}{\omega^2 - \omega_0^2} + B(t) \] (11)

We can solve (7) using Laplace transforms. If you’re unfamiliar with Laplace transforms, a good introduction can be found (at the time of writing - 3 Jun 2019) here. Most textbooks on differential equations or advanced calculus should also describe how to define and use them.

In practice, it’s easiest to use a table of Laplace transforms rather than work them all out from scratch, so we’ll do that here. To solve (7) we can think of this equation as an ordinary differential equation with independent variable \( t \) and constant parameter \( u \). The Laplace transform of a second derivative is given by

\[ \mathcal{L} \left( \partial_t^2 G(t, u) \right) = s^2 \Gamma(s) - s \mathcal{L} \left( G(0, u) \right) - \dot{G}(0, u) \] (12)

where \( \Gamma(s) \) is the Laplace transform of \( G(t, u) \). The variable \( s \) is the usual Laplace transform parameter, which will disappear when we convert the answer back at the end.

The given initial conditions in the problem are \( G(0, u) = \dot{G}(0, u) = 0 \), so we have

\[ \mathcal{L} \left( \partial_t^2 G(t, u) \right) = s^2 \Gamma(s) \] (13)

On the RHS of (7) we see from a table that the Laplace transform of a delta function is

\[ \mathcal{L} (\delta(t-u)) = e^{-us} \] (14)

Putting all this together we get the Laplace transform of (7).
\[ m \left( s^2 + \omega_0^2 \right) \Gamma(s) = e^{-us} \]  
\[ \Gamma(s) = \frac{e^{-us}}{m \left( s^2 + \omega_0^2 \right)} \]  

Again from a table, we see that we can write this as

\[ \frac{e^{-us}}{m \left( s^2 + \omega_0^2 \right)} = e^{-us} H(s) \]  

where

\[ H(s) = \frac{1}{m \left( s^2 + \omega_0^2 \right)} \]

An exponential \( e^{-us} \) multiplying a Laplace transform indicates an original function that is delayed by the amount \( u \). That is, if \( h(t) \) is the inverse transform of \( H(s) \), then the inverse transform of \( e^{-us} H(s) \) is \( h(t-u) \).

From tables, the inverse transform of \( H(s) \) is given by

\[ h(t) = \frac{1}{m\omega_0} \sin \omega_0 t \]

Thus the inverse transform of \( \Gamma(s) \) is

\[ G(t,u) = \frac{1}{m\omega_0} \sin \omega_0 (t-u) \]

Now suppose we have an explicit form of the forcing function given by

\[ f(t) = F_0 \sin \omega_0 t \]

The differential equation for this oscillator is therefore

\[ m \frac{\partial^2}{\partial t^2} A(t) + m\omega_0^2 A(t) = F_0 \sin \omega_0 t \]

With the Green function given by \( G(t,u) \) we can find \( A(t) \) by doing the integral

\[ A(t) = \int_0^t G(t,u) f(u) \, du \]

\[ = \frac{F_0}{m\omega_0} \int_0^t du \sin \omega_0 (t-u) \sin \omega_0 u \]

\[ = \frac{F_0}{2m\omega_0} \left( \sin \omega_0 t - \omega_0 t \cos \omega_0 t \right) \]
I used Maple to do the integral. If you want to do it by hand, you’ll need to expand $\sin \omega_0 (t - u)$ and then use some double angle formulas from trigonometry.