PROPAGATOR FOR FREE PARTICLE IN MOMENTUM SPACE

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Reference: Tom Lancaster and Stephen J. Blundell, *Quantum Field Theory for the Gifted Amateur*, (Oxford University Press, 2014), Problem 17.1.

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In section 17.1, L&B define the propagator for a particle in a general state $|\Omega\rangle$ to be

$$G^{+}(x,y)=\theta\left(x^{0}-y^{0}\right)\left\langle\Omega\left|\hat{\phi}(x)\hat{\phi}^{\dagger}(y)\right|\Omega\right\rangle \tag{1}$$

This is the probability amplitude that the system starts in state $|\Omega\rangle$, then at time y^0 a particle is created at position $\boldsymbol{y}$ and then at a later time x^0, the particle is annihilated at position $\boldsymbol{x}$. The step function $\theta\left(x^0-y^0\right)$ restricts x^0 to be at a later time than y^0. The field operator $\phi(x)$ (I'll leave off the hats from now on to save typing and clutter) is defined in the usual way in terms of creation and annihilation operators.

$$\phi(x)=\frac{1}{(2\pi)^{3/2}}\int\frac{d^{3}p}{\sqrt{2E_{p}}}\left(a(p)e^{-ip\cdot x}+a^{\dagger}(p)e^{ip\cdot x}\right) \tag{2}$$

In Exercise 17.1, L&B define the retarded field propagator for a free particle in momentum space as

$$G_{0}^{+}(\boldsymbol{p},t_{x},\boldsymbol{q},t_{y})=\theta(t_{x}-t_{y})\left\langle 0\left|a_{\boldsymbol{p}}(t_{x})a_{\boldsymbol{q}}^{\dagger}(t_{y})\right|0\right\rangle \tag{3}$$

It's not clear to me what a creation or annihilation operator is as a function of time, but I suspect what is meant is that a particle is created at time t_y with momentum $\boldsymbol{q}$. This particle then propagates to time t_x and is then subjected to annihilation if its momentum is then $\boldsymbol{p}$. The time evolution operator is the usual e^{-iHt}, where H is the hamiltonian, so we can write 3 as

$$G_{0}^{+}(\boldsymbol{p},t_{x},\boldsymbol{q},t_{y})=\theta(t_{x}-t_{y})\left\langle 0\left|a_{\boldsymbol{p}}e^{-iH(t_{x}-t_{y})}a_{\boldsymbol{q}}^{\dagger}\right|0\right\rangle \tag{4}$$

That is, a particle is just created by $a_{\boldsymbol{q}}^{\dagger}$, then evolves for a time t_x-t_y and is then annihilated by $a_{\boldsymbol{p}}$.

We can now use

$$a_{\boldsymbol{q}}^{\dagger}|0\rangle = |\boldsymbol{q}\rangle \tag{5}$$

$$\langle 0|\, a_{\boldsymbol{p}} = \langle \boldsymbol{p}| \tag{6}$$

$$\langle \boldsymbol{p}\,|\boldsymbol{q}\,\rangle = \delta\,(\boldsymbol{p}-\boldsymbol{q}) \tag{7}$$

Also, since the states $|\boldsymbol{p}\rangle$ and $|\boldsymbol{q}\rangle$ are states with a fixed momentum, they are energy eigenstates so operating on them with H results in repacing H with $E_{\boldsymbol{p}}$ or $E_{\boldsymbol{q}}$. Therefore we have

$$\left\langle 0\left|a_{\boldsymbol{p}}e^{-iH(t_x-t_y)}a_{\boldsymbol{q}}^{\dagger}\right|0\right\rangle = \left\langle 0\left|a_{\boldsymbol{p}}e^{-iE_{\boldsymbol{q}}t_x+iE_{\boldsymbol{q}}t_y}a_{\boldsymbol{q}}^{\dagger}\right|0\right\rangle \tag{8}$$

$$= e^{-iE_{\boldsymbol{q}}(t_x-t_y)}\,\langle \boldsymbol{p}\,|\boldsymbol{q}\,\rangle \tag{9}$$

$$= e^{-iE_{\boldsymbol{q}}(t_x-t_y)}\delta\,(\boldsymbol{p}-\boldsymbol{q}) \tag{10}$$

The delta function forces $\boldsymbol{p}=\boldsymbol{q}$, so this expression is equivalent to

$$G_0^+\,(\boldsymbol{p},t_x,\boldsymbol{q},t_y) = \theta\,(t_x-t_y)\,e^{-i(E_{\boldsymbol{p}}t_x-E_{\boldsymbol{q}}t_y)}\delta\,(\boldsymbol{p}-\boldsymbol{q}) \tag{11}$$

I'm not sure how we could justify splitting the exponential operator $e^{-iH(t_x-t_y)}$ so that the t_x part operates on the $\langle \boldsymbol{p}|$ to its left and the t_y part operates on the $|\boldsymbol{q}\rangle$ to its right, so I'm not entirely sure that this is the correct way of solving this problem. Comments welcome.

Comments

From Pedro Dardengo Mesquita. 13 Mar 2020, 17:55.

About the solution of the problem on the book QFT for the gifted amateur that asks you to derive the propagator for a free particle in momentum space. As a first comment I would like suggest to explicitly write the time dependence of the creation and annihilation operators, since they are given in this case as the standard time dependence in interaction picture. ($\hbar = 1$) $a(t) = e^{iHt}ae^{-iHt}$ and $a^{\dagger} = e^{-iHt}a^{\dagger}e^{iHt}$.

Also it would be interesting to specify the operator e^{-iHt} acting on the ground state of this non interacting system $|0\rangle$. Since there is no interaction the Hamiltonian is quadratic on the creation and annihilation operators. This makes the ground state $|0\rangle$ an eigenstate of the operator e^{-iHt}. So: $e^{-iHt}|0\rangle = e^{0}|0\rangle = |0\rangle$. The same happens in the other side of the expression when e^{iHt} acts on $\langle 0|$. This suggestion is merely to facilitate the comprehension of how the problem was solved.

In the last few lines you say you are not sure if the operator $e^{-iH(t_x-t_y)}$ can be split. They can be just like you did, since the Baker-Campbell-Hausdorff formula ensures this is valid the commutators $[Ht_x, Ht_y] = 0$ and $e^{-iH(t_x-t_y)} = e^{-iHt_x}e^{iHt_y}$.

Thank you for posting the solutions to these problems, they are helping me a lot.