GREEN FUNCTION FOR KLEIN-GORDON EQUATION

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In section 17.2, L&B derive an expression for the Feynman propagator, defined as

\[ \Delta(x, y) = \left\langle 0 \left| T \phi(x) \phi^\dagger(y) \right| 0 \right\rangle \]  

(1)

where \( T \) is the time ordering operator and \( \phi \) is the scalar field operator. Their derivation results in equation 17.15:

\[ \Delta(x, y) = \int \frac{d^3 p}{(2\pi)^3 2E_p} \left[ \theta(x^0 - y^0) e^{-ip \cdot (x-y)} + \theta(y^0 - x^0) e^{ip \cdot (x-y)} \right] \]  

(2)

Here \( \theta \) is the step function. Our task is to show that \( \Delta(x, y) \) fills the role of a Green function for the Klein-Gordon equation, so that it satisfies

\[ \left( \partial^2 + m^2 \right) \Delta(x, y) = -i\delta(x - y) \]  

(3)

where the derivatives on the LHS are with respect to components of \( x \), not \( y \).

Our task is made slightly easier since they require that we show this in a (1+1) dimensional system (one time dimension and one space dimension). That is, we have to show

\[ \left( \partial_0^2 - \partial_1^2 + m^2 \right) \Delta(x, y) = -i\delta(x^0 - y^0) \delta(x^1 - y^1) \]  

(4)

To take things one step at a time, we’ll split (2) into two parts:

\[ \Delta_1 = \int \frac{dp}{(2\pi)^3 2E_p} \theta(x^0 - y^0) e^{-ip \cdot (x-y)} \]  

(5)

\[ \Delta_2 = \int \frac{dp}{(2\pi)^3 2E_p} \theta(y^0 - x^0) e^{ip \cdot (x-y)} \]  

(6)

In what follows, we need the derivative of the step function.
\[ \partial_0 \theta (x^0 - y^0) \equiv \frac{\partial}{\partial x^0} \theta (x^0 - y^0) = \delta (x^0 - y^0) \tag{7} \]
\[ \frac{\partial}{\partial x^0} \theta (y^0 - x^0) = -\delta (y^0 - x^0) = -\delta (x^0 - y^0) \tag{8} \]

We also need the derivative of the delta function which satisfies the condition for any function \( f \):
\[ f (x) \delta' (x) = -f' (x) \delta (x) \tag{9} \]

For \( \Delta_1 \), we have
\[ \partial_0 \Delta_1 (x, y) = \int \frac{dp}{(2\pi)^2 E_p} \left[ \delta (x^0 - y^0) e^{-ip (x-\cdot)} - i E_p \theta (x^0 - y^0) e^{-ip (x-y)} \right] \tag{10} \]
\[ \partial_0^2 \Delta_1 (x, y) = \int \frac{dp}{(2\pi)^2 E_p} \left[ \delta' (x^0 - y^0) e^{-ip (x-\cdot)} - i E_p \delta (x^0 - y^0) e^{-ip (x-y)} \right. \]
\[ \left. -i E_p \delta (x^0 - y^0) e^{-ip (x-\cdot)} - E^2_p \theta (x^0 - y^0) e^{-ip (x-y)} \right] \tag{11} \]

Using \( \delta' (x^0 - y^0) e^{-ip (x-\cdot)} = i E_p \delta (x^0 - y^0) e^{-ip (x-y)} \) \( \tag{12} \)

Thus the first two terms in the brackets cancel and we’re left with
\[ \partial_0^2 \Delta_1 (x, y) = - \int \frac{dp}{(2\pi)^2 E_p} \left[ i E_p \delta (x^0 - y^0) e^{-ip (x-\cdot)} + E^2_p \theta (x^0 - y^0) e^{-ip (x-y)} \right] \tag{13} \]

We can do the same calculation on \( \Delta_2 \) and use \( \delta' (x^0 - y^0) e^{-ip (x-\cdot)} = i E_p \delta (x^0 - y^0) e^{-ip (x-y)} \) \( \tag{8} \)

The spatial derivatives are
\[ \partial_1^2 \Delta_1 (x, y) = - \int \frac{dp}{(2\pi)^2 E_p} \theta (x^0 - y^0) (p^1)^2 e^{-ip (x-y)} \tag{15} \]
\[ \partial_1^2 \Delta_2 (x, y) = - \int \frac{dp}{(2\pi)^2 E_p} \theta (y^0 - x^0) (p^1)^2 e^{ip (x-y)} \tag{16} \]

Finally, the \( m^2 \) terms are
\[ m^2 \Delta_1 (x, y) = \int \frac{dp}{(2\pi)^2} \frac{m^2 \theta (x^0 - y^0)}{2E_p} e^{-ip \cdot (x - y)} \]  

(17)

\[ m^2 \Delta_2 (x, y) = \int \frac{dp}{(2\pi)^2} \frac{m^2 \theta (y^0 - x^0)}{2E_p} e^{ip \cdot (x - y)} \]  

(18)

We can now use

\[ E^2_{pp} = (p^1)^2 + m^2 \]  

(19)

to combine the last term in (15) with (18) and (17) as specified on the LHS of (4):

\[
\int \frac{dp}{(2\pi)^2} \frac{E^2_{pp} - (p^1)^2 + m^2}{2E_p} \theta (x^0 - y^0) e^{-ip \cdot (x - y)} = 0
\]

(20)

We get a similar result with the components from \( \Delta_2 \):

\[
\int \frac{dp}{(2\pi)^2} \frac{E^2_{pp} - (p^1)^2 + m^2}{2E_p} \theta (y^0 - x^0) e^{ip \cdot (x - y)} = 0
\]

(21)

We’re therefore left with

\[
(\partial^2_0 - \partial^2_1 + m^2) \Delta (x, y) = - \int \frac{dp}{(2\pi)^2} \frac{E_p \delta (x^0 - y^0)}{2E_p} \left( e^{-ip \cdot (x - y)} + e^{ip \cdot (x - y)} \right)
\]

(22)

\[
= -i \delta (x^0 - y^0) \int \frac{dp^1}{2\pi} e^{-ip^1 \cdot (x^1 - y^1)}
\]

(23)

\[
= -i \delta (x^0 - y^0) \delta (x^1 - y^1)
\]

(24)

We’ve converted the exponent \( p \cdot (x - y) \) to just \( p^1 (x^1 - y^1) \) since the delta function \( \delta (x^0 - y^0) \) restricts \( x^0 = y^0 \). We’ve also used the standard definition of the delta function (in one dimension) as

\[
\delta (x^0 - y^0) = \frac{1}{2\pi} \int dp^1 e^{-ip^1 \cdot (x^1 - y^1)} = \frac{1}{2\pi} \int dp^1 e^{ip^1 \cdot (x^1 - y^1)}
\]

(25)

Note that it doesn’t matter which sign we choose in the exponent since we’re integrating over all \( p^1 \).