GREEN FUNCTION FOR KLEIN-GORDON EQUATION

Link to: physicspages home page.
To leave a comment or report an error, please use the auxiliary blog.
Reference: Tom Lancaster and Stephen J. Blundell, *Quantum Field Theory for the Gifted Amateur*, (Oxford University Press, 2014), Problem 17.2.
Post date: 15 Jun 2019.

In section 17.2, L&B derive an expression for the Feynman propagator, defined as

$$\Delta(x,y)=\left\langle 0\left|T\phi(x)\phi^{\dagger}(y)\right|0\right\rangle \tag{1}$$

where T is the time ordering operator and ϕ is the scalar field operator. Their derivation results in equation 17.15:

$$\Delta(x,y)=\int\frac{d^{3}p}{(2\pi)^{3}2E_{\boldsymbol{p}}}\left[\theta\left(x^{0}-y^{0}\right)e^{-ip\cdot(x-y)}+\theta\left(y^{0}-x^{0}\right)e^{ip\cdot(x-y)}\right] \tag{2}$$

Here θ is the step function. Our task is to show that $\Delta(x,y)$ fills the role of a Green function for the Klein-Gordon equation, so that it satisfies

$$\left(\partial^{2}+m^{2}\right)\Delta(x,y)=-i\delta(x-y) \tag{3}$$

where the derivatives on the LHS are with respect to components of x, not y.

Our task is made slightly easier since they require that we show this in a (1+1) dimensional system (one time dimension and one space dimension). That is, we have to show

$$\left(\partial_{0}^{2}-\partial_{1}^{2}+m^{2}\right)\Delta(x,y)=-i\delta\left(x^{0}-y^{0}\right)\delta\left(x^{1}-y^{1}\right) \tag{4}$$

To take things one step at a time, we'll split 2 into two parts:

$$\Delta_{1}=\int\frac{dp}{(2\pi)2E_{\boldsymbol{p}}}\theta\left(x^{0}-y^{0}\right)e^{-ip\cdot(x-y)} \tag{5}$$

$$\Delta_{2}=\int\frac{dp}{(2\pi)2E_{\boldsymbol{p}}}\theta\left(y^{0}-x^{0}\right)e^{ip\cdot(x-y)} \tag{6}$$

There is only a 2π in the denominator because we now have only one space dimension.

In what follows, we need the derivative of the step function:

$$\partial_0\theta\left(x^0-y^0\right)\equiv\frac{\partial}{\partial x^0}\theta\left(x^0-y^0\right)=\delta\left(x^0-y^0\right) \tag{7}$$

$$\frac{\partial}{\partial x^0}\theta\left(y^0-x^0\right)=-\delta\left(y^0-x^0\right)=-\delta\left(x^0-y^0\right) \tag{8}$$

We also need the derivative of the delta function, which satisfies the condition for any function f:

$$f(x)\,\delta'(x)=-f'(x)\,\delta(x) \tag{9}$$

For Δ_1, we have

$$\partial_0\Delta_1(x,y)=\int\frac{dp}{(2\pi)\,2E_{\boldsymbol{p}}}\left[\delta\left(x^0-y^0\right)e^{-ip\cdot(x-y)}-iE_{\boldsymbol{p}}\theta\left(x^0-y^0\right)e^{-ip\cdot(x-y)}\right] \tag{10}$$

$$\begin{aligned}\partial_0^2\Delta_1(x,y)=\int\frac{dp}{(2\pi)\,2E_{\boldsymbol{p}}}&\left[\delta'\left(x^0-y^0\right)e^{-ip\cdot(x-y)}-iE_{\boldsymbol{p}}\delta\left(x^0-y^0\right)e^{-ip\cdot(x-y)}\right.\\&\left.-iE_{\boldsymbol{p}}\delta\left(x^0-y^0\right)e^{-ip\cdot(x-y)}-E_{\boldsymbol{p}}^2\theta\left(x^0-y^0\right)e^{-ip\cdot(x-y)}\right]\end{aligned} \tag{11}$$

Using 9, the first term in the brackets is

$$\delta'\left(x^0-y^0\right)e^{-ip\cdot(x-y)}=iE_{\boldsymbol{p}}\delta\left(x^0-y^0\right)e^{-ip\cdot(x-y)} \tag{12}$$

Thus the first two terms in the brackets cancel and we're left with

$$\partial_0^2\Delta_1(x,y)=-\int\frac{dp}{(2\pi)\,2E_{\boldsymbol{p}}}\left[iE_{\boldsymbol{p}}\delta\left(x^0-y^0\right)e^{-ip\cdot(x-y)}+E_{\boldsymbol{p}}^2\theta\left(x^0-y^0\right)e^{-ip\cdot(x-y)}\right] \tag{13}$$

We can do the same calculation on Δ_2 and use 8 to get

$$\partial_0^2\Delta_2(x,y)=-\int\frac{dp}{(2\pi)\,2E_{\boldsymbol{p}}}\left[iE_{\boldsymbol{p}}\delta\left(x^0-y^0\right)e^{ip\cdot(x-y)}+E_{\boldsymbol{p}}^2\theta\left(y^0-x^0\right)e^{ip\cdot(x-y)}\right] \tag{14}$$

The spatial derivatives are

$$\partial_1^2\Delta_1(x,y)=-\int\frac{dp}{(2\pi)\,2E_{\boldsymbol{p}}}\theta\left(x^0-y^0\right)\left(p^1\right)^2e^{-ip\cdot(x-y)} \tag{15}$$

$$\partial_1^2\Delta_2(x,y)=-\int\frac{dp}{(2\pi)\,2E_{\boldsymbol{p}}}\theta\left(y^0-x^0\right)\left(p^1\right)^2e^{ip\cdot(x-y)} \tag{16}$$

Finally, the m^2 terms are

$$m^2\Delta_1(x,y) = \int \frac{dp}{(2\pi)\,2E_{\boldsymbol{p}}} m^2\theta\left(x^0-y^0\right)e^{-ip\cdot(x-y)} \tag{17}$$

$$m^2\Delta_2(x,y) = \int \frac{dp}{(2\pi)\,2E_{\boldsymbol{p}}} m^2\theta\left(y^0-x^0\right)e^{ip\cdot(x-y)} \tag{18}$$

We can now use

$$E_{\boldsymbol{p}}^2 = \left(p^1\right)^2 + m^2 \tag{19}$$

to combine the last term in 13 with 15 and 17 as specified on the LHS of 4:

$$\int \frac{dp}{(2\pi)\,2E_{\boldsymbol{p}}}\left(-E_{\boldsymbol{p}}^2+\left(p^1\right)^2+m^2\right)\theta\left(x^0-y^0\right)e^{-ip\cdot(x-y)} = 0 \tag{20}$$

We get a similar result with the components from Δ_2:

$$\int \frac{dp}{(2\pi)\,2E_{\boldsymbol{p}}}\left(-E_{\boldsymbol{p}}^2+\left(p^1\right)^2+m^2\right)\theta\left(y^0-x^0\right)e^{ip\cdot(x-y)} = 0 \tag{21}$$

We're therefore left with

$$\left(\partial_0^2-\partial_1^2+m^2\right)\Delta(x,y) = -\int \frac{dp}{(2\pi)\,2E_{\boldsymbol{p}}}\left(iE_{\boldsymbol{p}}\delta\left(x^0-y^0\right)\left(e^{-ip\cdot(x-y)}+e^{ip\cdot(x-y)}\right)\right) \tag{22}$$

$$= -i\delta\left(x^0-y^0\right)\int\frac{dp^1}{2\pi}e^{-ip^1\left(x^1-y^1\right)} \tag{23}$$

$$= -i\delta\left(x^0-y^0\right)\delta\left(x^1-y^1\right) \tag{24}$$

We've converted the exponent $p\cdot(x-y)$ to just $p^1\left(x^1-y^1\right)$ since the delta function $\delta\left(x^0-y^0\right)$ restricts $x^0=y^0$. We've also used the standard definition of the delta function (in one dimension) as

$$\delta\left(x^0-y^0\right) = \frac{1}{2\pi}\int dp^1 e^{-ip^1\left(x^1-y^1\right)} = \frac{1}{2\pi}\int dp^1 e^{ip^1\left(x^1-y^1\right)} \tag{25}$$

Note that it doesn't matter which sign we choose in the exponent since we're integrating over all p^1.