ACTION OF A SCALAR FIELD THEORY AS A MOMENTUM INTEGRAL

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We’ll look at a scalar field theory defined by the Lagrangian density:

\[
L = \frac{1}{2} \partial_\mu \phi(x) \partial^\mu \phi(x) - \frac{m^2}{2} [\phi(x)]^2
\]  

(1)

The field operator \( \phi(x) \) can also be written as the Fourier transform of a field operator in momentum space:

\[
\phi(x) = \frac{1}{(2\pi)^4} \int d^4 p \ e^{ip \cdot x} \tilde{\phi}(p)
\]  

(2)

In this form, the Lagrangian density is

\[
\mathcal{L} = \frac{1}{2(2\pi)^8} \int d^4 p \ d^4 q \ e^{i(p+q) \cdot x} (\phi(p) \tilde{\phi}(q) - p_\mu q^\mu - m^2)
\]  

(3)

The action is the integral of the Lagrangian density over all space and time, so we have

\[
S = \int d^4 x \ L
\]  

(4)

\[
= \frac{1}{2(2\pi)^8} \int d^4 x \int d^4 p \ d^4 q \ e^{i(p+q) \cdot x} (\phi(p) \tilde{\phi}(q) - p_\mu q^\mu - m^2)
\]  

(5)

The integral over spacetime gives us a delta function according to the formula

\[
\delta(p + q) = \frac{1}{(2\pi)^4} \int d^4 x \ e^{i(p+q) \cdot x}
\]  

(6)

We get

\[
S = \frac{1}{2(2\pi)^4} \int d^4 p \ d^4 q \ \delta(p + q) (\phi(p) \tilde{\phi}(q) - m^2)
\]  

(7)

The delta function thus sets \( q = -p \) so we can do the integral over \( q \) to get
\[ S = \frac{1}{2(2\pi)^4} \int d^4 p \ (p_\mu p^\mu - m^2) \tilde{\phi}(p) \tilde{\phi}(-p) \]  

(8)

\[ = \frac{1}{(2\pi)^4} \int d^4 p \left( \frac{p^2 - m^2}{2} \right) \tilde{\phi}(p) \tilde{\phi}(-p) \]  

(9)

L&B add at the end of the problem that we can read off the propagator by taking the inverse of the term in parentheses and multiplying by \( i/2 \). That is, we take the factor

\[ \frac{p^2 - m^2}{2} \]  

invert it and multiply by \( i/2 \) (and add in an infinitesimal imaginary part to prevent blowups).

I’m not sure why this is true, but if we do this, we get

\[ \tilde{G}(p) = \frac{i}{p^2 - m^2 + i\epsilon} \]  

(11)

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