PROPAGATORS FOR A COUPLE OF DISCRETIZED SYSTEMS

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Reference: Tom Lancaster and Stephen J. Blundell, *Quantum Field Theory for the Gifted Amateur*, (Oxford University Press, 2014), Problem 17.5.

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We can use the technique developed for the Klein-Gordon propagator to find a propagator for a couple of other examples.

Example 1. First we consider the one-dimensional system with Lagrangian density

$$\mathcal{L}=\frac{1}{2}\left(\frac{\partial\phi(x)}{\partial x}\right)^2+\frac{m^2}{2}[\phi(x)]^2 \tag{1}$$

This is the same Lagrangian as for the Klein-Gordon field except the second term now has a plus sign. We discretize the system by putting it on a lattice, where the field is now defined at a sequence of N lattice points, equally spaced by distance a. The Fourier expansion is now

$$\phi_j=\frac{1}{\sqrt{Na}}\sum_p\tilde{\phi}_p e^{ipja} \tag{2}$$

where the index j labels the lattice points and runs from 1 to N.

To use this in the Lagrangian 1, we need the derivative which is now a finite difference:

$$\begin{aligned}\frac{\partial\phi_j}{\partial x}&=\frac{\phi_{j+1}-\phi_j}{a} && (3)\\ &=\frac{1}{a\sqrt{Na}}\sum_p\tilde{\phi}_p e^{ipja}\left(e^{ipa}-1\right) && (4)\end{aligned}$$

The Lagrangian therefore becomes

$$\mathcal{L}_j = \frac{1}{2Na^3}\sum_p\sum_q \tilde{\phi}_p\tilde{\phi}_q e^{ipja}\left(e^{ipa}-1\right)e^{iqja}\left(e^{iqa}-1\right)+ \tag{5}$$

$$\frac{m^2}{2Na}\sum_p\sum_q \tilde{\phi}_p\tilde{\phi}_q e^{ipja}e^{iqja} \tag{6}$$

$$= \frac{1}{2Na}\sum_p\sum_q \tilde{\phi}_p\tilde{\phi}_q \left[e^{i(p+q)ja}\left(\frac{1}{a^2}\left(e^{ipa}-1\right)\left(e^{iqa}-1\right)+m^2\right)\right] \tag{7}$$

The action S is the sum of $\mathcal{L}$ over all the lattice points, where we must multiply each increment in the sum by the lattice spacing a, so we get

$$S = \sum_j a\mathcal{L}_j \tag{8}$$

From 7 we see that the only term depending on j is $e^{i(p+q)ja}$. We can use the identity (I remember this being true, although I can't find a reference at the moment):

$$\sum_{j=1}^{N} e^{i(p+q)ja} = N\delta_{q,-p} \tag{9}$$

Using this, we have

$$S = \frac{1}{2}\sum_p \tilde{\phi}_p\tilde{\phi}_{-p}\left[\frac{1}{a^2}\left(e^{-ipa}-1\right)\left(e^{ipa}-1\right)+m^2\right] \tag{10}$$

The term involving the exponentials can be simplified:

$$\left(e^{-ipa}-1\right)\left(e^{ipa}-1\right) = 1-\left(e^{ipa}+e^{-ipa}\right)+1 \tag{11}$$

$$= 2-2\cos pa \tag{12}$$

Thus we have

$$S = \frac{1}{2}\sum_p \tilde{\phi}_p\tilde{\phi}_{-p}\left(\frac{1}{a^2}(2-2\cos pa)+m^2\right) \tag{13}$$

Applying the recipe for extracting the propagator we get

$$\tilde{G}(p) = \frac{i}{(2-2\cos pa)/a^2+m^2} \tag{14}$$

Example 2. We now consider a Lagrangian of form

$$\mathcal{L} = \frac{1}{2}\left[(\partial_0\phi)^2-(\partial_1\phi)^2\right] \tag{15}$$

Again, we discretize the spatial dimension by writing the Fourier expansion

$$\phi_j(t) = \frac{1}{\sqrt{Na}} \sum_p \int \frac{d\omega}{2\pi} \tilde{\phi}_p(\omega) e^{-i\omega t} e^{ipja} \tag{16}$$

The time derivative is

$$\partial_0 \phi_j(t) = \frac{-i\omega}{\sqrt{Na}} \sum_p \int \frac{d\omega}{2\pi} \tilde{\phi}_p(\omega) e^{-i\omega t} e^{ipja} \tag{17}$$

The space derivative is a finite difference, as above:

$$\partial_1 \phi_j(t) = \frac{\phi_{j+1} - \phi_j}{a} \tag{18}$$

$$= \frac{1}{a\sqrt{Na}} \sum_p \int \frac{d\omega}{2\pi} \tilde{\phi}_p(\omega) e^{-i\omega t} e^{ipja} \left(e^{ipa} - 1\right) \tag{19}$$

The action is

$$S = \sum_j a \int dt\ \mathcal{L} \tag{20}$$

$$= -\frac{a}{2Na(2\pi)^2} \sum_{j,p,q} \int d\omega d\omega' dt\ \omega\omega' \tilde{\phi}_p(\omega) \tilde{\phi}_q(\omega') e^{-i(\omega+\omega')t} e^{i(p+q)ja} - \frac{a}{2Na^3(2\pi)^2} \sum_{j,p,q} \int d\omega d\omega' dt\ \tilde{\phi}_p(\omega) \tilde{\phi}_q(\omega') e^{-i(\omega+\omega')t} e^{i(p+q)ja} \left(e^{-ipa} - 1\right)\left(e^{iqa} - 1\right) \tag{21}$$

The integral over t uses one of the 2π factors in the denominator and gives a $\delta(\omega + \omega')$ so we get

$$S = \frac{1}{2N(2\pi)} \sum_{j,p,q} \int d\omega\ \omega^2 \tilde{\phi}_p(\omega) \tilde{\phi}_q(-\omega) e^{i(p+q)ja} - \frac{1}{2Na^2(2\pi)^2} \sum_{j,p,q} \int d\omega\ \tilde{\phi}_p(\omega) \tilde{\phi}_q(-\omega) e^{i(p+q)ja} \left(e^{-ipa} - 1\right)\left(e^{iqa} - 1\right) \tag{22}$$

The sum over j uses 9 and we get

$$S = \frac{1}{2\cdot 2\pi}\sum_p \int d\omega \tilde{\phi}_p(\omega)\,\tilde{\phi}_q(-\omega)\left[\omega^2 - \frac{1}{a^2}\left(e^{-ipa}-1\right)\left(e^{ipa}-1\right)\right] \quad (23)$$

$$= \frac{1}{2\cdot 2\pi}\sum_p \int d\omega \tilde{\phi}_p(\omega)\,\tilde{\phi}_q(-\omega)\left[\omega^2 - \frac{2}{a^2}(1-\cos pa)\right] \quad (24)$$

Applying the recipe for extracting the propagator we get

$$\tilde{G}(\omega,p) = \frac{i}{\omega^2 - \frac{2}{a^2}(1-\cos pa)} \quad (25)$$

$$= \frac{i}{\omega^2 - \omega_0^2(1-\cos pa)} \quad (26)$$

where $\omega_0^2 = 2/a^2$.