WICK'S THEOREM - EXAMPLES

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Reference: Tom Lancaster and Stephen J. Blundell, *Quantum Field Theory for the Gifted Amateur*, (Oxford University Press, 2014), Problems 18.3 - 18.5.

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We'll look at a few examples of Wick's theorem, which is used to express a time-ordered product as an expansion in terms of normal ordered products. L&B define a contraction of two operators as the difference between a time-ordered product and normal-ordered product of these operators. That is

$$\underbracket{AB} = T[AB] - N[AB] \tag{1}$$

L&B use a top bracket for a contraction, but to be consistent with earlier posts, I'll use a bottom bracket.

where T indicates time-ordering and N indicates normal ordering. In practice, a contraction is equivalent to a commutator of the two operators, as shown in L&B's equation 18.27:

$$\underbracket{A(x)B}(y) = \begin{cases} \left[A^-(x), B^+(y)\right] & \text{if } x^0 > y^0 \\ \left[B^-(y), A^+(x)\right] & \text{if } x^0 < y^0 \end{cases} \tag{2}$$

Here, A^+ represents that portion of the field operator containing creation operators and A^- the portion containing annihilation operators. If this formula is applied to bare creation or annihilation operators, it says that the contraction of two annihilation (or of two creation) operators is zero.

In these three problems, L&B don't say explicitly that we're dealing with time-ordered products, but I assume we must be as otherwise Wick's theorem doesn't apply.

Example 1. Apply Wick's theorem to the string of Bose operators $a_{\boldsymbol{p}}a_{\boldsymbol{q}}^{\dagger}a_{\boldsymbol{k}}$. In this case, the only non-zero contractions are those involving $a_{\boldsymbol{q}}^{\dagger}$ and one of the annihilation operators, so we have

$$T\left[a_{\boldsymbol{p}}a_{\boldsymbol{q}}^{\dagger}a_{\boldsymbol{k}}\right] = N\left[a_{\boldsymbol{p}}a_{\boldsymbol{q}}^{\dagger}a_{\boldsymbol{k}}\right] + \underbracket{a_{\boldsymbol{p}}a_{\boldsymbol{q}}^{\dagger}}a_{\boldsymbol{k}} + \underbracket{a_{\boldsymbol{q}}^{\dagger}a_{\boldsymbol{k}}}a_{\boldsymbol{p}} \tag{3}$$

The commutator of two Bose operators is

$$\left[a_{\boldsymbol{p}}, a_{\boldsymbol{q}}^{\dagger}\right] = \delta(\boldsymbol{p}-\boldsymbol{q}) \tag{4}$$

I'm not quite sure what to make of the time-dependence in this case, as usually the time-dependence of a field operator is found in the exponentials in the mode expansion and not in the creation or annihilation operators. If we just apply the commutator directly we get

$$T\left[a_{\boldsymbol{p}} a_{\boldsymbol{q}}^{\dagger} a_{\boldsymbol{k}}\right] = a_{\boldsymbol{q}}^{\dagger} a_{\boldsymbol{p}} a_{\boldsymbol{k}} + a_{\boldsymbol{k}} \delta\left(\boldsymbol{p}-\boldsymbol{q}\right) + a_{\boldsymbol{p}} \delta\left(\boldsymbol{q}-\boldsymbol{k}\right) \tag{5}$$

Example 2. We're given a string of Bose operators: $bgbb^{\dagger}b^{\dagger}$ and asked to normal-order this using Bose commutators. This isn't strictly 'normal ordering' since in normal ordering we just swap operators to get all the creation operators on the left, so we'd get

$$N\left[bgbb^{\dagger}b^{\dagger}\right] = b^{\dagger}b^{\dagger}bgb \tag{6}$$

If we move the $b^{\dagger}$s to the left using commutators, we are essentially rewriting the original expression in an equivalent form. To do this, we use

$$\left[b, b^{\dagger}\right] = bb^{\dagger} - b^{\dagger}b = 1 \tag{7}$$

with all other commutators being zero. (I'm assuming that the g operator is just an annihilation operator for a particle different from b.) We get

$$bgbb^{\dagger}b^{\dagger} = b\left(1+b^{\dagger}b\right)b^{\dagger}g \tag{8}$$

$$= bb^{\dagger}g + bb^{\dagger}bb^{\dagger}g \tag{9}$$

$$= \left(1+b^{\dagger}b\right)g + \left(1+b^{\dagger}b\right)\left(1+b^{\dagger}b\right)g \tag{10}$$

$$= 2g + 3b^{\dagger}bg + b^{\dagger}bb^{\dagger}bg \tag{11}$$

$$= 2g + 3b^{\dagger}bg + b^{\dagger}\left(1+b^{\dagger}b\right)bg \tag{12}$$

$$= 2g + 4b^{\dagger}bg + b^{\dagger}b^{\dagger}bbg \tag{13}$$

Applying Wick's theorem, and using the fact that the only non-zero contraction is

$$\underbracket{bb^{\dagger}} = \left[b, b^{\dagger}\right] = 1 \tag{14}$$

we get

$$T\left[bgbb^{\dagger}b^{\dagger}\right] = N\left[bgbb^{\dagger}b^{\dagger}\right] + 2\underbracket{bb^{\dagger}}\,\underbracket{bb^{\dagger}}g + 4\underbracket{bb^{\dagger}}b^{\dagger}bg \tag{15}$$

$$= b^{\dagger}b^{\dagger}bbg + 2g + 4b^{\dagger}bg \tag{16}$$

The second term on the RHS comes from contracting the first b with the first $b^{\dagger}$ and the second b with the second $b^{\dagger}$, and then contracting the first b with the second $b^{\dagger}$ and the second b with the first $b^{\dagger}$, giving $2\underbracket{bb^{\dagger}}\,\underbracket{bb^{\dagger}}g$. The last term comes from the 4 ways of contracting one of the bs with one of the $b^{\dagger}$s. Thus the two methods give the same answer.

Example 3. We now have a vacuum expectation value (VEV) of fermion operators:

$$\left\langle 0\left|c^{\dagger}_{\mathbf{p}_1-\mathbf{q}}c^{\dagger}_{\mathbf{p}_2+\mathbf{q}}c_{\mathbf{p}_2}c_{\mathbf{p}_1}\right|0\right\rangle \tag{17}$$

Applying Wick's theorem gives us a sum of terms with either no contractions, one contraction or two contractions. If we have any uncontracted operators, the VEV of that term will be zero, so the only non-zero contribution comes from terms with two contractions (fully contracted). Using the result 18.36 in the text, we then have

$$\begin{aligned}\left\langle 0\left|c^{\dagger}_{\mathbf{p}_1-\mathbf{q}}c^{\dagger}_{\mathbf{p}_2+\mathbf{q}}c_{\mathbf{p}_2}c_{\mathbf{p}_1}\right|0\right\rangle = & -\left\langle 0\left|T\left[c^{\dagger}_{\mathbf{p}_1-\mathbf{q}}c_{\mathbf{p}_2}\right]\right|0\right\rangle\left\langle 0\left|T\left[c^{\dagger}_{\mathbf{p}_2+\mathbf{q}}c_{\mathbf{p}_1}\right]\right|0\right\rangle + \\ & \left\langle 0\left|T\left[c^{\dagger}_{\mathbf{p}_1-\mathbf{q}}c_{\mathbf{p}_1}\right]\right|0\right\rangle\left\langle 0\left|T\left[c^{\dagger}_{\mathbf{p}_2+\mathbf{q}}c_{\mathbf{p}_2}\right]\right|0\right\rangle\end{aligned} \tag{18}$$

The minus sign on the first term arises from swapping the $c_{\mathbf{p}_2}$ with the $c^{\dagger}_{\mathbf{p}_2+\mathbf{q}}$ in order to form the first contraction. This single swap of two fermion operators introduces a sign change. The second term requires two swaps to bring $c_{\mathbf{p}_1}$ next to $c^{\dagger}_{\mathbf{p}_1-\mathbf{q}}$, so the two sign changes cancel out.