MOMENTUM SPACE AMPLITUDES FOR PHI-FOUR THEORY

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Reference: Tom Lancaster and Stephen J. Blundell, *Quantum Field Theory for the Gifted Amateur*, (Oxford University Press, 2014), Problems 19.1, 19.4.

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In Chapter 19, L&B consider the system governed by the Lagrangian

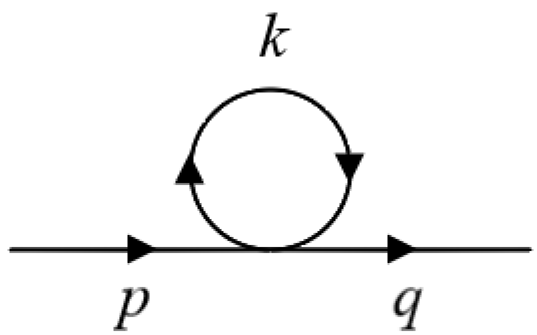
$$\mathcal{L}=\frac{1}{2}\left[\partial_{\mu}\phi(x)\right]^{2}-\frac{m^{2}}{2}\phi(x)^{2}-\frac{\lambda}{4!}\phi(x)^{4} \tag{1}$$

Here, λ is a constant parameter. The interaction is contained in the last term involving ϕ^4, so this system is known as "phi-four". In their Fig 19.6, they show a number of Feynman diagrams arising from this theory. Five of these diagrams involve just a single particle, while the other four involve two particles scattering off each other. Our task is to write down the amplitudes for each of these diagrams in momentum space, where we can use the rules stated in the text. These are

- Each vertex contributes a factor $-i\lambda$.
- Each internal line (that is, a line with both ends connected) is labelled with a momentum (q, say) and is described by a propagator term of form $\frac{i}{q^2-m^2+i\epsilon}$.
- Conserve momentum at each vertex by requiring the total momentum entering a vertex to be equal to the total momentum leaving that vertex.
- Integrate over each internal momentum with an integration measure of $\frac{d^4q}{(2\pi)^4}$.
- Each external line (with one end unconnected) contributes a factor of 1.
- Divide by the symmetry factor (worked out in their Example 19.5) D.
- Multiply by an overall delta function (and another factor of $(2\pi)^4$) to conserve energy-momentum in the process as a whole.

First, we consider the single-particle diagrams.

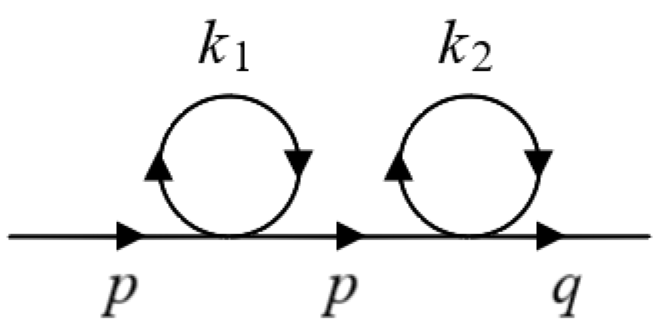
Example 1. Single loop.



There is a single vertex, with one internal line (a loop with momentum k). The symmetry factor for a loop is $D=2$, because the loop can be flipped about its vertical axis to give the same result, so there are two equivalent diagrams. The delta function must ensure that $p=q$, conserving energy-momentum. The amplitude is thus

$$-(2\pi)^4\delta^{(4)}(p-q)\frac{i\lambda}{2}\int\frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\epsilon} \tag{2}$$

Example 2. Double loop.

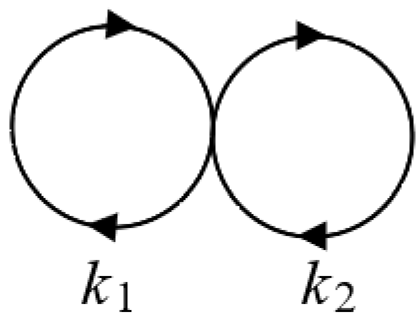


We now have two vertices and 3 internal lines. Due to momentum conservation the middle horizontal line must have the same momentum as the incoming line. The momentum of each loop can vary so must be integrated over. The symmetry factor here is $D=2\times2=4$, since we have a factor of 2 for each loop. We get

$$(2\pi)^4\delta^{(4)}(p-q)\frac{(-i\lambda)^2}{4}\frac{i}{p^2-m^2+i\epsilon}\int\frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\frac{i}{k_1^2-m^2+i\epsilon}\frac{i}{k_2^2-m^2+i\epsilon} \tag{3}$$

The middle horizontal line contributes the factor $\frac{i}{p^2-m^2+i\epsilon}$, but as p is constrained to be equal to the incoming momentum, it is not integrated.

Example 3. Isolated double loop (a vacuum diagram, since both loops arise out of the vacuum without any incoming particles).



We have a single vertex and two internal lines. The symmetry factor is $2\times 2=4$ for the two loops, but we can also swap the positions of the two loops giving another factor of 2, so $D=8$. There is no incoming or outgoing particle. Applying the rules gives us

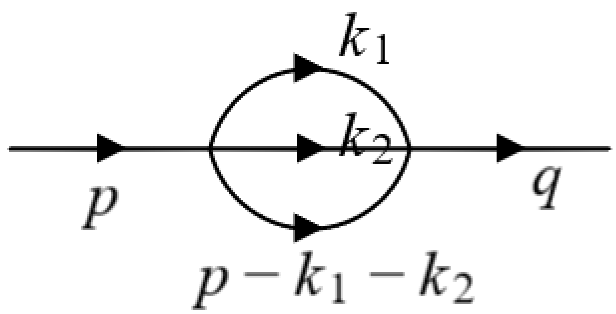
$$\frac{-i\lambda}{8}\int\frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\frac{i}{k_1^2-m^2+i\epsilon}\frac{i}{k_2^2-m^2+i\epsilon}=-\frac{i\lambda}{8}\left[\int\frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\epsilon}\right]^2 \tag{4}$$

What about the delta function that would normally appear to conserve momentum? I'm not entirely sure of the derivation, but if we look at Example 19.6 where L&B go through the details of calculating a momentum space amplitude, we see that the delta function arises from an integral of form

$$\int d^4w\; e^{iq\cdot w}e^{-ip\cdot w} \tag{5}$$

where the integral over the spacetime variable w is over the volume and time of interest in the experiment. If $p\neq q$, this gives us $(2\pi)^4\delta^{(4)}(p-q)$. In the vacuum case, I suppose we could argue that both p and q are zero, which gives us the infinite delta function $\delta^{(4)}(0)$, which is what L&B quote in their exercise. Since this condition applies to any diagram without an incoming or outgoing particle, it would apply to any vacuum case. Comments welcome.

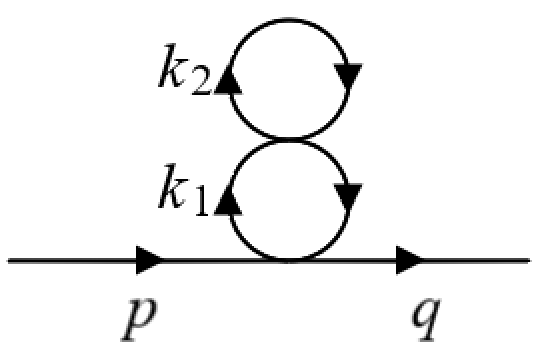
Example 4. Saturn diagram.



Here we have 2 vertices and 3 internal lines. The 3 internal lines can be permuted in $3!=6$ ways, so $D=6$. One of these internal lines must be constrained to conserve momentum, so I've labelled the bottom line $p-k_1-k_2$. The amplitude is

$$(2\pi)^4\delta^{(4)}(p-q)\frac{(-i\lambda)^2}{6}\int\frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\frac{i}{k_1^2-m^2+i\epsilon}\frac{i}{k_2^2-m^2+i\epsilon}\frac{i}{(p-k_1-k_2)^2-m^2+i\epsilon} \tag{6}$$

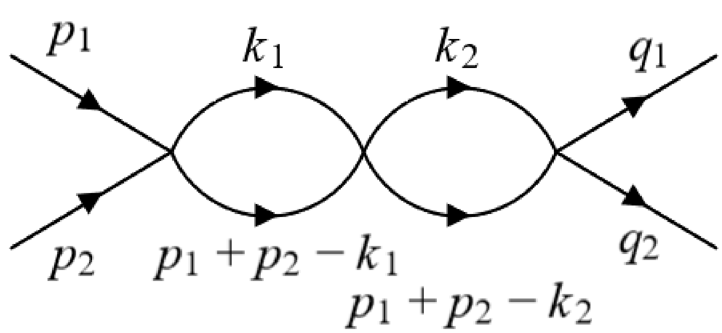
Example 5. Double loop with incoming particle.



There are 2 vertices and 3 internal lines, since each branch of the lower loop contributes an internal line. Conservation of momentum requires that the momentum of the lower loop is the same on both sides, since at the top vertex, a momentum k_2 leaves and returns to the same point, so there is no net momentum change at this vertex. The symmetry factor is $D = 2 \times 2 = 4$. The amplitude is

$$(2\pi)^4 \delta^{(4)}(p-q) \frac{(-i\lambda)^2}{4} \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \frac{i}{k_1^2 - m^2 + i\epsilon} \frac{i}{k_2^2 - m^2 + i\epsilon} \quad (7)$$

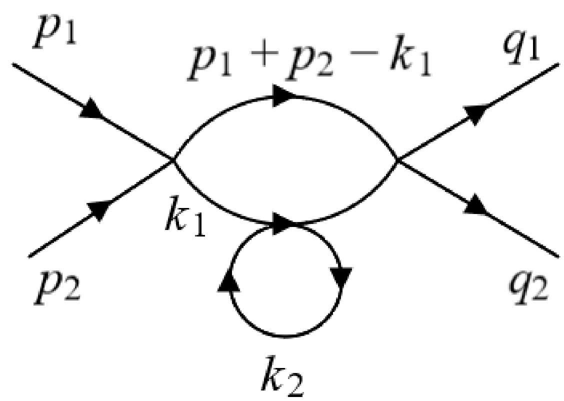
Example 6. Scattering with four branches.



We have 3 vertices and 4 internal lines, where we've labelled the lines using momentum conservation. The symmetry factor is $D = 2 \times 2 = 4$, since each pair of internal lines can be swapped. The overall momentum conserving delta function is now $\delta^{(4)}(p_1 + p_2 - q_1 - q_2)$, so the amplitude is

$$(2\pi)^4 \delta^{(4)}(p_1 + p_2 - q_1 - q_2) \frac{(-i\lambda)^3}{4} \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \frac{i}{k_1^2 - m^2 + i\epsilon} \frac{i}{k_2^2 - m^2 + i\epsilon} \times \frac{i}{(p_1 + p_2 - k_1)^2 - m^2 + i\epsilon} \frac{i}{(p_1 + p_2 - k_2)^2 - m^2 + i\epsilon} \quad (8)$$

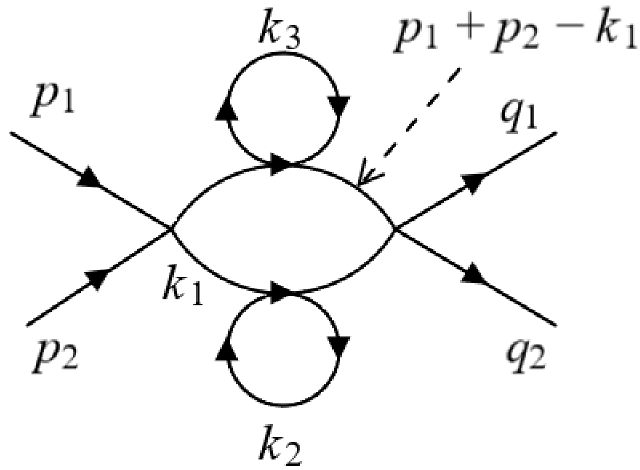
Example 7. Scattering with internal loop.



There are 3 vertices and 4 internal lines. The symmetry factor is $D = 2$, since the only symmetry is that the loop at the bottom can be flipped. The amplitude is

$$(2\pi)^4 \delta^{(4)}(p_1+p_2-q_1-q_2)\frac{(-i\lambda)^3}{2}\int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\left[\frac{i}{k_1^2-m^2+i\epsilon}\right]^2\frac{i}{k_2^2-m^2+i\epsilon}\times \frac{i}{(p_1+p_2-k_1)^2-m^2+i\epsilon} \tag{9}$$

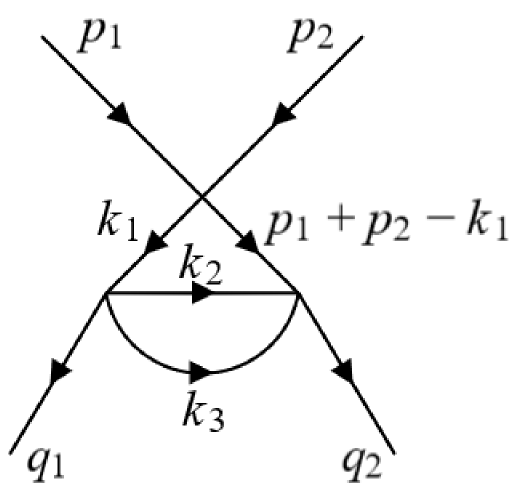
Example 8. Scattering with two internal loops.



We have 4 vertices and 6 internal lines. The symmetry factor is $D = 8$, since we can flip each of the two loops and also swap the upper and lower branches. The amplitude is

$$(2\pi)^4 \delta^{(4)}(p_1+p_2-q_1-q_2) \frac{(-i\lambda)^4}{8} \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \frac{d^4k_3}{(2\pi)^4} \left[\frac{i}{k_1^2-m^2+i\epsilon}\right]^2 \frac{i}{k_2^2-m^2+i\epsilon} \times \left[\frac{i}{(p_1+p_2-k_1)^2-m^2+i\epsilon}\right]^2 \frac{i}{k_3^2-m^2+i\epsilon} \tag{10}$$

Example 9. Complex scattering.



I'm not certain which direction we're supposed to interpret this diagram, so I've taken the incoming particles to be at the top. We then have 3 vertices and 4 internal lines. The symmetry factor is $D=2$ since only the k_2 and k_3 lines can be swapped. The amplitude is

$$(2\pi)^4\delta^{(4)}(p_1+p_2-q_1-q_2)\frac{(-i\lambda)^3}{2}\int\frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\frac{d^4k_3}{(2\pi)^4}\frac{i}{k_1^2-m^2+i\epsilon}\frac{i}{k_2^2-m^2+i\epsilon}\times \frac{i}{k_3^2-m^2+i\epsilon}\frac{i}{(p_1+p_2-k_1)^2-m^2+i\epsilon} \tag{11}$$

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