FOURIER TRANSFORM OF YUKAWA POTENTIAL

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In calculating the scattering cross-section in Yukawa’s $\psi^\dagger \psi \phi$ theory we come across the Fourier transform of a potential, as given in L&B’s equation 20.25:

$$\tilde{V}(q) = -\frac{g^2}{|q|^2 + \mu^2}$$

We are asked to show that the Fourier transform of the Yukawa potential

$$V(r) = -\frac{g^2}{4\pi r} e^{-\mu r}$$

is indeed $1$. We can do this by just evaluating the integral. The Fourier transform is

$$\tilde{V}(q) = \int dr \, d\theta \, d\phi \, r^2 \sin \theta e^{iqr} V(r)$$

$$= -\frac{g^2}{4\pi} \int_0^\infty \int_0^\pi \int_0^{2\pi} dr \, d\theta \, d\phi \, r \sin \theta e^{iqr} \cos \theta e^{-\mu r}$$

We can do this integral directly using Maple, and we do in fact get the result $1$. If you want to do it by hand, it’s not too hard. We have (I’m using $r = |r|$ and $q = |q|$):

1
\[
\int_0^\infty \int_0^\pi \int_0^{2\pi} dr \, d\theta \, d\phi \, r \sin \theta e^{iqr \cos \theta} e^{-\mu r} = 2\pi \int_0^\infty \int_0^\pi dr \, r \sin \theta e^{iqr \cos \theta} e^{-\mu r} \\
= \frac{2\pi i}{q} \int_0^\infty e^{-\mu r} \left[ e^{iqr \cos \theta} \right]_{\theta=\pi}^{\theta=0} \\
= -2\pi i \left( e^{2iqr} - 1 \right) e^{-iqr - \mu r} \\
= -2\pi i \left( e^{iqr - \mu r} - \frac{e^{-iqr - \mu r}}{iq - \mu} \right) \\
= 2\pi \left( -q + i\mu \right) e^{i\mu + q} - e^{-r(iq + \mu)} (i\mu + q) \bigg|_{r=0}^{r=\infty} \\
= -\frac{4\pi}{q^2 + \mu^2} \\
\]

We can use this result to get the Fourier transform of the Coulomb potential. In the SI system, this potential is

\[V(\mathbf{r}) = \frac{Q}{4\pi \varepsilon_0 r}\]

where \(Q\) is the charge. This is the limit of \(2\) as \(\mu \to 0\), with \(g^2 = -Q/\varepsilon_0\) so we have

\[\tilde{V}(\mathbf{q}) = \frac{Q}{\varepsilon_0 q^2} \]

In Gaussian units, the Coulomb potential is

\[V(\mathbf{r}) = \frac{Q}{r}\]

so the Fourier transform is

\[\tilde{V}(\mathbf{q}) = \frac{4\pi Q}{q^2} \]