FOURIER TRANSFORM OF YUKAWA POTENTIAL

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Reference: Tom Lancaster and Stephen J. Blundell, *Quantum Field Theory for the Gifted Amateur*, (Oxford University Press, 2014), Problem 20.2.

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In calculating the scattering cross-section in Yukawa's $\psi^{\dagger}\psi\phi$ theory we come across the Fourier transform of a potential, as given in L&B's equation 20.25:

$$\tilde{V}(\boldsymbol{q}) = -\frac{g^2}{|\boldsymbol{q}|^2 + \mu^2} \tag{1}$$

We are asked to show that the Fourier transform of the Yukawa potential

$$V(\boldsymbol{r}) = -\frac{g^2}{4\pi r}e^{-\mu r} \tag{2}$$

is indeed 1. We can do this by just evaluating the integral. The Fourier transform is

$$\tilde{V}(\boldsymbol{q}) = \int dr\, d\theta\, d\phi\, r^2 \sin\theta e^{i\boldsymbol{q}\cdot\boldsymbol{r}} V(\boldsymbol{r}) \tag{3}$$

$$= -\frac{g^2}{4\pi}\int_0^{\infty}\int_0^{\pi}\int_0^{2\pi} dr\, d\theta\, d\phi\, r\sin\theta e^{iqr\cos\theta}e^{-\mu r} \tag{4}$$

We can do this integral directly using Maple, and we do in fact get the result 1. If you want to do it by hand, it's not too hard. We have (I'm using $r = |\boldsymbol{r}|$ and $q = |\boldsymbol{q}|$):

$$\int_0^\infty \int_0^\pi \int_0^{2\pi} dr\, d\theta\, d\phi\, r\sin\theta e^{iqr\cos\theta} e^{-\mu r} = 2\pi \int_0^\infty \int_0^\pi dr\, d\theta\, r\sin\theta e^{iqr\cos\theta} e^{-\mu r} \tag{5}$$

$$= \frac{2\pi i}{q} \int_0^\infty e^{-\mu r} \left[e^{iqr\cos\theta} \right]_{\theta=0}^{\theta=\pi} \tag{6}$$

$$= \frac{-2\pi i \left(\mathrm{e}^{2\,iqr} - 1\right) \mathrm{e}^{-iqr-\mu r}}{q} \tag{7}$$

$$= -\frac{2\pi i}{q} \left(\frac{\mathrm{e}^{iqr-\mu r}}{iq-\mu} - \frac{\mathrm{e}^{-iqr-\mu r}}{-iq-\mu} \right)_{r=0}^{r=\infty} \tag{8}$$

$$= 2\pi \left. \frac{(-q+i\mu)\,\mathrm{e}^{r(iq-\mu)} - \mathrm{e}^{-r(iq+\mu)}\,(i\mu+q)}{q\left(\mu^2+q^2\right)} \right|_{r=0}^{r=\infty} \tag{9}$$

$$= -\frac{4\pi}{q^2+\mu^2} \tag{10}$$

We can use this result to get the Fourier transform of the Coulomb potential. In the SI system, this potential is

$$V(\boldsymbol{r}) = \frac{Q}{4\pi\epsilon_0 r} \tag{11}$$

where Q is the charge. This is the limit of 2 as $\mu \to 0$, with $g^2 = -Q/\epsilon_0$ so we have

$$\tilde{V}(\boldsymbol{q}) = \frac{Q}{\epsilon_0 q^2} \tag{12}$$

In Gaussian units, the Coulomb potential is

$$V(\boldsymbol{r}) = \frac{Q}{r} \tag{13}$$

so the Fourier transform is

$$\tilde{V}(\boldsymbol{q}) = \frac{4\pi Q}{q^2} \tag{14}$$