RESPONSE FUNCTION FOR FORCED HARMONIC OSCILLATOR

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Reference: Tom Lancaster and Stephen J. Blundell, *Quantum Field Theory for the Gifted Amateur*, (Oxford University Press, 2014), Problem 21.2.

Post date: 30 Jul 2019.

We consider the forced harmonic oscillator with Lagrangian

$$L=\frac{1}{2}m\dot{x}^{2}(t)-\frac{1}{2}m\omega^{2}x^{2}(t)+f(t)x(t) \tag{1}$$

The response function $\chi(t-t')$ is defined by

$$\langle\psi(t)|x(t)|\psi(t)\rangle=\int_{-\infty}^{\infty}dt'\chi\left(t-t'\right)f\left(t'\right) \tag{2}$$

The expectation value of the position of the oscillator is given by the expression on the LHS, and it's defined so that it depends on the forcing function for all times prior to the current time t. In other words, the current position depends on the entire history of the forcing function.

We can get an approximate formula for χ by using the interaction picture, in which we split the hamiltonian into two parts with H_0 representing the non-interaction part and H' the interaction part. In our case

$$H'=-f(t)x(t) \tag{3}$$

and the interaction hamiltonian is given by

$$H_I(t)=e^{iH_0t}H'e^{-iH_0t} \tag{4}$$

and the interaction state is given by

$$|\psi_I(t)\rangle=e^{iH_0t}|\psi(t)\rangle \tag{5}$$

In the problem statement, L&B refer to $f_I(t)$ and $x_I(t)$, which I'm assuming are defined in the same way as $H_I(t)$ by sandwiching the original function or operator between exponentials of H_0. In that case, we have

$$\begin{aligned}H_I(t)&=-e^{iH_0t}f(t)e^{-iH_0t}e^{iH_0t}x(t)e^{-iH_0t} &(6)\\&=-f_I(t)x_I(t) &(7)\end{aligned}$$

The state vector evolves using the evolution operator U_I according to

$$\left|\psi_I(t)\right\rangle = U_I(t,-\infty)\left|0\right\rangle \tag{8}$$

where we're assuming the oscillator starts in the ground state at $t=-\infty$. The evolution operator has the form

$$U_I(t,-\infty) = T\left[e^{-i\int_{-\infty}^{t}dt'H_I(t')}\right] \tag{9}$$

where T indicates time-ordering. We can expand this up to first order to get

$$U_I(t,-\infty) = 1-i\int_{-\infty}^{t}dt'H_I\left(t'\right) \tag{10}$$

$$= 1+i\int_{-\infty}^{t}dt'f_I\left(t'\right)x_I\left(t'\right) \tag{11}$$

so the state at time t is, from 8

$$\left|\psi_I(t)\right\rangle = \left|0\right\rangle + i\int_{-\infty}^{t}dt'f_I\left(t'\right)x_I\left(t'\right)\left|0\right\rangle \tag{12}$$

From here, we can work out the LHS of 2. First we have

$$x(t) = e^{-iH_0t}x_I(t)e^{iH_0t} \tag{13}$$

so we have, using 5

$$\left\langle\psi(t)\left|x(t)\right|\psi(t)\right\rangle = \left\langle\psi(t)\left|e^{-iH_0t}x_I(t)e^{iH_0t}\right|\psi(t)\right\rangle \tag{14}$$

$$= \left\langle\psi_I(t)\left|x_I(t)\right|\psi_I(t)\right\rangle \tag{15}$$

We can now use 12 to get, again to first order:

$$\left\langle\psi(t)\left|x(t)\right|\psi(t)\right\rangle = \left\langle 0\left|x_I(t)\right|0\right\rangle - i\left\langle 0\left|\int_{-\infty}^{t}dt'f_I\left(t'\right)x_I\left(t'\right)x_I(t)\right|0\right\rangle$$

$$+i\left\langle 0\left|\int_{-\infty}^{t}dt'x_I(t)f_I\left(t'\right)x_I\left(t'\right)\right|0\right\rangle \tag{16}$$

$$= \left\langle 0\left|x_I(t)\right|0\right\rangle + i\left\langle 0\left|\int_{-\infty}^{t}dt'\left[x_I(t),x_I\left(t'\right)\right]f_I\left(t'\right)\right|0\right\rangle \tag{17}$$

In the last term, the integration variable t' is always less than t (since the upper limit of integration is t), so we can write this as

$$\langle\psi(t)|x(t)|\psi(t)\rangle = \langle 0|x_I(t)|0\rangle + i\left\langle 0\left|\int_{-\infty}^{\infty} dt'\theta\left(t-t'\right)\left[x_I(t), x_I\left(t'\right)\right] f_I\left(t'\right)\right|0\right\rangle \tag{18}$$

To deal with the first term, we use the expression for the position operator in terms of the creation and annihilation operators

$$x(t) = \frac{1}{\sqrt{2m\omega}}\left(a + a^{\dagger}\right) \tag{19}$$

The interaction version is then obtained by assuming that the creation operator evolves according to $a^{\dagger}(t) = e^{i\omega t}a^{\dagger}$ so we get

$$x_I(t) = \frac{1}{\sqrt{2m\omega}}\left(ae^{-i\omega t} + a^{\dagger}e^{i\omega t}\right) \tag{20}$$

Because we can express x_I in terms of a creation and an annihilation operator, the term $\langle 0|x_I(t)|0\rangle$ will be zero, since the annihilation operator acting to the right on $|0\rangle$ and the creation operator acting to the left on $\langle 0|$ both give zero. Therefore we have

$$\langle\psi(t)|x(t)|\psi(t)\rangle = i\left\langle 0\left|\int_{-\infty}^{\infty} dt'\theta\left(t-t'\right)\left[x_I(t), x_I\left(t'\right)\right] f_I\left(t'\right)\right|0\right\rangle \tag{21}$$

By comparing this with 2 we find

$$\chi\left(t-t'\right) = i\theta\left(t-t'\right)\langle 0|\left[x_I(t), x_I\left(t'\right)\right]|0\rangle \tag{22}$$

From 20 and the commutator

$$\left[a, a^{\dagger}\right] = 1 \tag{23}$$

we can work out the commutator:

$$\left[x_I(t), x_I\left(t'\right)\right] = \frac{1}{2m\omega}\left[\left(aa^{\dagger} + a^{\dagger}a\right)e^{-i\omega(t-t')} - \left(a^{\dagger}a + aa^{\dagger}\right)e^{i\omega(t-t')}\right] \tag{24}$$

$$= \frac{1}{2m\omega}\left[\left(1 + 2a^{\dagger}a\right)e^{-i\omega(t-t')} - \left(1 + 2a^{\dagger}a\right)e^{i\omega(t-t')}\right] \tag{25}$$

$$= \frac{1}{2m\omega}(1+2n)\left[e^{-i\omega(t-t')} - e^{i\omega(t-t')}\right] \tag{26}$$

$$= -\frac{2i}{m\omega}\left(n + \frac{1}{2}\right)\sin\omega\left(t-t'\right) \tag{27}$$

where $n=a^{\dagger}a$ is the number operator. The response function is thus

$$\chi\left(t-t'\right)=\theta\left(t-t'\right)\left\langle 0\left|\frac{2}{m\omega}\left(n+\frac{1}{2}\right)\sin\omega\left(t-t'\right)\right|0\right\rangle \tag{28}$$

Since $\langle 0|n|0\rangle=0$ (there are no particles in the ground state), this reduces to

$$\chi\left(t-t'\right)=\theta\left(t-t'\right)\frac{\sin\omega\left(t-t'\right)}{m\omega} \tag{29}$$

In the next section, L&B ask us to work out χ for a temperature $T>0$. In this case, we won't be in the ground state any more, so I'm assuming that we take the average of χ over all states $|n\rangle$ rather than just using the vacuum state $|0\rangle$. We can then use the result from earlier:

$$\langle n\rangle_t=\frac{1}{e^{\beta\omega}-1} \tag{30}$$

which gives

$$\chi\left(t-t'\right)=\theta\left(t-t'\right)\frac{2}{m\omega}\left(\frac{1}{e^{\beta\omega}-1}+\frac{1}{2}\right)\sin\omega\left(t-t'\right) \tag{31}$$

The correlation functon S is defined in terms of the anti-commutator as

$$S=\frac{1}{2}\left\langle\left\{x_I\left(t'\right),x_I(t)\right\}\right\rangle \tag{32}$$

Any operator with a different number of annihilation operators from creation operators will give zero when operating on any state $|n\rangle$ (since we'll always be trying to annihilate more operators than we've created, either by operating to the left or right). Thus using the anticommutator simply replaces the minus sign in 24 with a plus sign, so we have

$$\left\{x_I(t),x_I\left(t'\right)\right\}=\frac{1}{2m\omega}\left[\left(aa^{\dagger}+a^{\dagger}a\right)e^{-i\omega(t-t')}+\left(a^{\dagger}a+aa^{\dagger}\right)e^{i\omega(t-t')}\right] \tag{33}$$

$$=\frac{1}{2m\omega}\left[\left(1+2a^{\dagger}a\right)e^{-i\omega(t-t')}+\left(1+2a^{\dagger}a\right)e^{i\omega(t-t')}\right] \tag{34}$$

$$=\frac{1}{2m\omega}(1+2n)\left[e^{-i\omega(t-t')}+e^{i\omega(t-t')}\right] \tag{35}$$

$$=\frac{2}{m\omega}\left(n+\frac{1}{2}\right)\cos\omega\left(t-t'\right) \tag{36}$$

This gives us

$$S = \frac{1}{m\omega}\left(\frac{1}{e^{\beta\omega}-1}+\frac{1}{2}\right)\cos\omega\left(t-t'\right) \tag{37}$$