GREEN FUNCTION FOR DIFFUSION EQUATION

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Reference: Tom Lancaster and Stephen J. Blundell, *Quantum Field Theory for the Gifted Amateur*, (Oxford University Press, 2014), Problem 21.3.

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The diffusion or heat equation is

$$\frac{\partial n(\boldsymbol{x},t)}{\partial t}-D\nabla^{2}n(\boldsymbol{x},t)=0 \tag{1}$$

where $n(\boldsymbol{x},t)$ is the density of molecules (for diffusion) or temperature (for heat) at spatial location $\boldsymbol{x}$ at time t, and D is the coefficient of diffusion. In the general case, D could depend on $\boldsymbol{x}$ and t, but we'll take it to be a constant here. The diffusion equation can be augmented by adding a forcing term in place of the 0 on the RHS, but again, we'll omit that here.

The Green function $G(\boldsymbol{x}-\boldsymbol{y},t-u)$ for this equation is defined by the equation

$$\frac{\partial G}{\partial t}-D\nabla^{2}G=\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})\,\delta(t-u) \tag{2}$$

We are given an initial condition

$$G(\boldsymbol{x}-\boldsymbol{y},t=0)=\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}) \tag{3}$$

which corresponds to a point source at position $\boldsymbol{y}$ at $t=0$.

We can get the energy-momentum Green function by using a Fourier transform of G:

$$G(\boldsymbol{x}-\boldsymbol{y},t-u)=\frac{1}{(2\pi)^{4}}\int d^{3}q\;d\omega e^{-i\boldsymbol{q}\cdot(\boldsymbol{x}-\boldsymbol{y})}e^{-i\omega t}G(\omega,\boldsymbol{q}) \tag{4}$$

Plugging this into the LHS of 2 we have

$$\frac{\partial G}{\partial t}-D\nabla^{2}G=\frac{1}{(2\pi)^{4}}\int d^{3}q\;d\omega\left(-i\omega+Dq^{2}\right)e^{-i\boldsymbol{q}\cdot(\boldsymbol{x}-\boldsymbol{y})}e^{-i\omega t}G(\omega,\boldsymbol{q}) \tag{5}$$

In order for this to equal the RHS of 2, we therefore have

$$G(\omega,\boldsymbol{q})=\frac{1}{-i\omega+Dq^{2}} \tag{6}$$

I'm not sure I understand the point of the exercise as stated in L&B's book, since they ask us to derive 6 while taking into account the initial condition, but we haven't used the initial condition 3. As I don't have access to the book by Chaikin and Lubensky that they refer to in the question, I'll settle for deriving the Green function $G(\boldsymbol{x}-\boldsymbol{y},t)$ (with $u=0$) that satisfies 3.

To do this, we'll use a Fourier transform of $G(\boldsymbol{x}-\boldsymbol{y},t)$ in space only, leaving the time coordinate untouched. That is, we consider

$$G(\boldsymbol{x}-\boldsymbol{y},t)=\frac{1}{(2\pi)^3}\int d^3q\, e^{i\boldsymbol{q}\cdot(\boldsymbol{x}-\boldsymbol{y})}G(\boldsymbol{q},t) \tag{7}$$

Plugging this into the LHS of 2 we have

$$\frac{\partial G}{\partial t}-D\nabla^2G=\frac{1}{(2\pi)^3}\int d^3q\, e^{i\boldsymbol{q}\cdot(\boldsymbol{x}-\boldsymbol{y})}\left(\dot{G}(\boldsymbol{q},t)+Dq^2G(\boldsymbol{q},t)\right) \tag{8}$$

In order for this to match the RHS of 2 with $u=0$ we therefore must have

$$\dot{G}(\boldsymbol{q},t)+Dq^2G(\boldsymbol{q},t)=\delta(t) \tag{9}$$

We can solve this differential equation by using an integrating factor. That is, we multiply both sides by e^{Dq^2t} and then observe that

$$\frac{\partial}{\partial t}\left(e^{Dq^2t}G(\boldsymbol{q},t)\right)=e^{Dq^2t}\left(\dot{G}(\boldsymbol{q},t)+Dq^2G(\boldsymbol{q},t)\right) \tag{10}$$

Therefore

$$\frac{\partial}{\partial t}\left(e^{Dq^2t}G(\boldsymbol{q},t)\right)=e^{Dq^2t}\delta(t) \tag{11}$$

We can now integrate both sides to get

$$e^{Dq^2t}G(\boldsymbol{q},t)=\int dt\, e^{Dq^2t}\delta(t) \tag{12}$$

The RHS is zero if the range of integration doesn't include $t=0$ (because of the delta function) and 1 otherwise (since $e^{Dq^2t}=1$ when $t=0$). Therefore, assuming we start the integration at $t=0$, we have, for $t>0$

$$G(\boldsymbol{q},t)=e^{-Dq^2t} \tag{13}$$

The Green function $G(\boldsymbol{x}-\boldsymbol{y},t)$ can now be found by taking the inverse Fourier transform (I did the integral using Maple. Doing it by hand is possible, though a bit messy):

$$G(\boldsymbol{x}-\boldsymbol{y},t) = \int d^3q\, e^{-i\boldsymbol{q}\cdot(\boldsymbol{x}-\boldsymbol{y})} e^{-Dq^2t} \tag{14}$$

$$= \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \int_0^{\infty} dq\, q^2 \sin\theta e^{-iq(x-y)\cos\theta} e^{-Dq^2t} \tag{15}$$

$$= \frac{\pi^{3/2}}{(Dt)^{3/2}} e^{-(\boldsymbol{x}-\boldsymbol{y})^2/4Dt} \tag{16}$$

If $\boldsymbol{x}=\boldsymbol{y}$ then this function goes to ∞ as $t\to 0$. If $\boldsymbol{x}\neq\boldsymbol{y}$, then the exponent goes to $-\infty$ as $t\to 0$, so the overall function goes to zero, since the exponential $e^{-(\boldsymbol{x}-\boldsymbol{y})^2/4Dt}$ goes to zero faster than the denominator $t^{3/2}$. Thus $G(\boldsymbol{x}-\boldsymbol{y},t)$ behaves like $\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})\,\delta(t)$.

We can check the normalization by doing the integral. For the purposes of the integral, we define a new spatial variable $\boldsymbol{r}\equiv\boldsymbol{x}-\boldsymbol{y}$, so we have (again, using Maple to do the integral):

$$\int d^3r\, G(\boldsymbol{x}-\boldsymbol{y},t) = \frac{\pi^{3/2}}{(Dt)^{3/2}} \int d^3r\, e^{-r^2/4Dt} \tag{17}$$

$$= (2\pi)^2 \tag{18}$$

In order for $G(\boldsymbol{x}-\boldsymbol{y},t)$ to be exactly a delta function, the normalization should be 1. However, I can't see where the missing factor of $(2\pi)^2$ is - comments welcome.