GREEN FUNCTION FOR DIFFUSION EQUATION

The diffusion or heat equation is

\[ \frac{\partial n(x,t)}{\partial t} - D \nabla^2 n(x,t) = 0 \]  

(1)

where \( n(x,t) \) is the density of molecules (for diffusion) or temperature (for heat) at spatial location \( x \) at time \( t \), and \( D \) is the coefficient of diffusion. In the general case, \( D \) could depend on \( x \) and \( t \), but we’ll take it to be a constant here. The diffusion equation can be augmented by adding a forcing term in place of the 0 on the RHS, but again, we’ll omit that here.

The Green function \( G(x-y,t-u) \) for this equation is defined by the equation

\[ \frac{\partial G}{\partial t} - D \nabla^2 G = \delta^{(3)}(x-y) \delta(t-u) \]  

(2)

We are given an initial condition

\[ G(x-y,t=0) = \delta^{(3)}(x-y) \]  

(3)

which corresponds to a point source at position \( y \) at \( t = 0 \).

We can get the energy-momentum Green function by using a Fourier transform of \( G \):

\[ G(x-y,t-u) = \frac{1}{(2\pi)^4} \int d^3 q \, d \omega e^{-iq \cdot (x-y)} e^{-i\omega t} G(\omega,q) \]  

(4)

Plugging this into the LHS of (2) we have

\[ \frac{\partial G}{\partial t} - D \nabla^2 G = \frac{1}{(2\pi)^4} \int d^3 q \, d \omega \left( -i\omega + Dq^2 \right) e^{-iq \cdot (x-y)} e^{-i\omega t} G(\omega,q) \]  

(5)

In order for this to equal the RHS of (2), we therefore have

\[ G(\omega,q) = \frac{1}{-i\omega + Dq^2} \]  

(6)
I’m not sure I understand the point of the exercise as stated in L&B’s book, since they ask us to derive [6] while taking into account the initial condition, but we haven’t used the initial condition [3]. As I don’t have access to the book by Chaikin and Lubensky that they refer to in the question, I’ll settle for deriving the Green function \( G(x-y,t) \) (with \( u = 0 \)) that satisfies [3].

To do this, we’ll use a Fourier transform of \( G(x-y,t) \) in space only, leaving the time coordinate untouched. That is, we consider

\[
G(x-y,t) = \frac{1}{(2\pi)^3} \int d^3q \ e^{iq \cdot (x-y)} G(q,t)
\]

Plugging this into the LHS of [2] we have

\[
\frac{\partial G}{\partial t} - D\nabla^2 G = \frac{1}{(2\pi)^3} \int d^3q \ e^{iq \cdot (x-y)} \left( \dot{G}(q,t) + Dq^2 G(q,t) \right)
\]

In order for this to match the RHS of [2] with \( u = 0 \) we therefore must have

\[
\dot{G}(q,t) + Dq^2 G(q,t) = \delta(t)
\]

We can solve this differential equation by using an integrating factor. That is, we multiply both sides by \( e^{Dq^2 t} \) and then observe that

\[
\frac{\partial}{\partial t} \left( e^{Dq^2 t} G(q,t) \right) = e^{Dq^2 t} \left( \dot{G}(q,t) + Dq^2 G(q,t) \right)
\]

Therefore

\[
\frac{\partial}{\partial t} \left( e^{Dq^2 t} G(q,t) \right) = e^{Dq^2 t} \delta(t)
\]

We can now integrate both sides to get

\[
e^{Dq^2 t} G(q,t) = \int dt \ e^{Dq^2 t} \delta(t)
\]

The RHS is zero if the range of integration doesn’t include \( t = 0 \) (because of the delta function) and 1 otherwise (since \( e^{Dq^2 t} = 1 \) when \( t = 0 \)). Therefore, assuming we start the integration at \( t = 0 \), we have, for \( t > 0 \)

\[
G(q,t) = e^{-Dq^2 t}
\]

The Green function \( G(x-y,t) \) can now be found by taking the inverse Fourier transform (I did the integral using Maple. Doing it by hand is possible, though a bit messy):
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\begin{equation}
G(x - y, t) = \int d^3q \ e^{-iq(x-y)} e^{-Dq^2t}
\end{equation}

\begin{equation}
= \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \int_0^{\infty} dq \ q^2 \sin \theta e^{-iq(x-y)\cos \theta} e^{-Dq^2t}
\end{equation}

\begin{equation}
= \frac{\pi^{3/2}}{(Dt)^{3/2}} e^{-(x-y)^2/4Dt}
\end{equation}

If \( x = y \) then this function goes to \( \infty \) as \( t \to 0 \). If \( x \neq y \), then the exponent goes to \( -\infty \) as \( t \to 0 \), so the overall function goes to zero, since the exponential \( e^{-(x-y)^2/4Dt} \) goes to zero faster than the denominator \( t^{3/2} \). Thus \( G(x - y, t) \) behaves like \( \delta^{(3)}(x - y) \delta(t) \).

We can check the normalization by doing the integral. For the purposes of the integral, we define a new spatial variable \( r \equiv x - y \), so we have (again, using Maple to do the integral):

\begin{equation}
\int d^3r \ G(x - y, t) = \frac{\pi^{3/2}}{(Dt)^{3/2}} \int d^3r \ e^{-r^2/4Dt}
\end{equation}

\begin{equation}
= (2\pi)^2
\end{equation}

In order for \( G(x - y, t) \) to be exactly a delta function, the normalization should be 1. However, I can’t see where the missing factor of \((2\pi)^2\) is - comments welcome.