GENERATING FUNCTIONAL FOR HARMONIC OSCILLATOR

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Reference: Tom Lancaster and Stephen J. Blundell, *Quantum Field Theory for the Gifted Amateur*, (Oxford University Press, 2014), Problem 22.1.

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In Chapter 22, L&B describe the method of using a generating functional to derive Green functions from either the S-matrix of Feynman diagrams. A lot of this is still rather unclear to me, so I'll just accept their results and leave a deeper explanation to a time when I have a deeper understanding.

In this problem, we consider the forced harmonic oscillator with Lagrangian

$$L=\frac{1}{2}m\dot{x}^{2}(t)-\frac{1}{2}m\omega^{2}x^{2}(t)+f(t)x(t) \tag{1}$$

Here $f(t)$ is the forcing function which has the constant value f_0 for $0\le t\le T$ and zero elsewhere. If we treat $f(t)x(t)$ as the interaction term, it is the analogue of the source term J used by L&B. The quantity $\langle 0|S|0\rangle_f$ is therefore the amplitude for starting with zero particles (or zero oscillator excitations in this case) at some past time (before $t=0$) and ending up with a system that also contains no particles (after $t=T$, when the forcing function has switched off). The amplitude $\langle 0|S|0\rangle_f$ contains all the vacuum Feynman diagrams (diagrams with no external lines, so they represent loops where particles spontaneously create and destroy themselves), and all diagrams where particles are created and annihilated by the forcing term, but with no net particle creation so we always end up with zero particles. According to L&B's equation 22.20, this is the same thing as the generating functional $Z[f(t)]$ so we have

$$Z[f(t)]=\langle 0|S|0\rangle_f \tag{2}$$

The S-matrix is given by Dyson's exponential:

$$S=T\left[e^{-i\int d^4x\ H_I(x)}\right] \tag{3}$$

where H_I is the interaction hamiltonian, which in our case is

$$H_I=f(t)x(t) \tag{4}$$

FIGURE 1. Dumbbell diagram

The expansion of S must contain terms with only even numbers of factors, since creation and annihilation must always occur in pairs. In terms of a Feynman diagram, this consists of the 'dumbbell' diagram where the two end blobs represent either particle creation or annihilation and the path joining them represents the propagator (Fig. 1).

The dumbbell diagram has an amplitude given by

$$\text{(Dumbbell)} = \frac{(-i)^2}{2}\int dt\, dt' f(t)\left\langle 0\left|Tx(t)\,x\left(t'\right)\right|0\right\rangle f\left(t'\right) \tag{5}$$

This follows by applying the Feynman rules for this system, which are similar to those we encountered earlier for ϕ^4 theory. In this case, since the scalar factor multiplying the interaction term $f(t)\,x(t)$ is just 1, each vertex contributes a factor of $-if$. The internal line contributes the propagator $\langle 0|Tx(t)\,x(t')|0\rangle$ and since we can swap the two vertices without changing the topology of the diagram, the symmetry factor is 2. The generating functional is the sum of all powers of this diagram where each term in the sum is divided by the symmetry factor $n!$, which is equivalent to the exponential of the dumbbell diagram. (The calculation is similar to that in L&B's equation 22.19). Thus

$$Z[f(t)] = e^{\text{(Dumbbell)}} \tag{6}$$

In momentum space, we need the Fourier transform of the forcing function, which is

$$\tilde{f}(\nu) = \int_{-\infty}^{\infty} dt\, e^{i\nu t} f(t) \tag{7}$$

$$= \int_0^T dt\, e^{i\nu t} f_0 \tag{8}$$

$$= \frac{f_0}{i\nu}\left(e^{i\nu T}-1\right) \tag{9}$$

$$= \frac{if_0}{\nu}\left(1-e^{i\nu T}\right) \tag{10}$$

The internal line is represented by the propagator for the harmonic oscillator which is

$$\tilde{G}(\nu)=\frac{i}{m(\nu^2-\omega^2+i\epsilon)} \tag{11}$$

We integrate over the internal momentum with an integration measure of $\frac{d\nu}{2\pi}$. Because of conservation of momentum, the $f(\nu)$ factor due to one of the vertices must be balanced by a $f(-\nu)$ factor at the other, so we end up with

$$\text{(Dumbbell)}=\frac{(-i)^2}{4\pi m}\int d\nu\, f(-\nu)\frac{i}{m(\nu^2-\omega^2+i\epsilon)}f(\nu) \tag{12}$$

$$=-\frac{1}{4\pi m}\int d\nu\, f(-\nu)\frac{i}{m(\nu^2-\omega^2+i\epsilon)}f(\nu) \tag{13}$$

The quantity $|Z[f(t)]|^2=|\langle 0|S|0\rangle|^2$ is the probability of starting and ending with zero excitations (particles). We can work it out by evaluating the integral in 13. To do this we use the suggestion in the question, where it is stated that

$$\frac{i}{E+i\epsilon}=\mathcal{P}\frac{1}{E}-i\pi\delta(E) \tag{14}$$

The symbol $\mathcal{P}$ indicates the principal value. I'm not entirely clear on this point, but I think what is implied is that we need to find the principal value of the integral

$$\mathcal{P}\int_{-E_0}^{E_0}\frac{dE}{E} \tag{15}$$

If we try to do this integral directly, it diverges around $E=0$ since $\ln E$ diverges here. The principal value of an improper integral such as this avoids the singularity by integrating up to some infinitesimal value on either side of $E=0$, working out the integral, and then taking the limit as $E\to 0$. In this case, because $\frac{1}{E}$ is an odd function, its integral over any pair of intervals symmetrically placed on either side of $E=0$ will be zero. This remains true no matter how close to $E=0$ we take the infinitestimal value, so the principal value of $\frac{1}{E}$ is zero.

With

$$E=\nu^2-\omega^2 \tag{16}$$

we have therefore

$$-\frac{1}{4\pi m}\int d\nu\ f(-\nu)\frac{i}{m(\nu^2-\omega^2+i\epsilon)}f(\nu)=\frac{1}{4\pi m}\int d\nu\ f(-\nu)\,i^2\pi\delta(E)\,f(\nu) \tag{17}$$

$$=-\frac{1}{4m}\int d\nu\ f(-\nu)\,\delta(E)\,f(\nu) \tag{18}$$

To evaluate this, we first work out the product of the two f functions, using 10:

$$f(-\nu)\,f(\nu)=\frac{if_0}{-\nu}\left(1-e^{-i\nu T}\right)\frac{if_0}{\nu}\left(1-e^{i\nu T}\right) \tag{19}$$

$$=\frac{f_0^2}{\nu^2}(2-2\cos\nu T) \tag{20}$$

$$=\frac{4f_0^2}{\nu^2}\sin^2\frac{\nu T}{2} \tag{21}$$

where in the last line we used the half-angle trig identity

$$\sin\frac{\theta}{2}=\sqrt{\frac{1-\cos\theta}{2}} \tag{22}$$

Next, we need to transform the delta function $\delta(E)$ in 18, which we can do using the formula

$$\delta(g(x))=\sum_i\frac{\delta(x-z_i)}{|g'(z_i)|} \tag{23}$$

where the z_i are the zeroes of $g(x)$. In our case, we have $E(\nu)=\nu^2-\omega^2$, which has two zeroes, at $\nu=\pm\omega$. At both these points, $|E'(\pm\omega)|=2\omega$, so we have

$$\delta(E)=\frac{1}{2\omega}(\delta(\nu-\omega)+\delta(\nu+\omega)) \tag{24}$$

Plugging this and 21 into 18 we have

$$-\frac{1}{4m}\int d\nu\ f(-\nu)\,\delta(E)\,f(\nu)=-\frac{4f_0^2}{8m\omega}\int d\nu\,(\delta(\nu-\omega)+\delta(\nu+\omega))\frac{\sin^2\frac{\nu T}{2}}{\nu^2} \tag{25}$$

$$=-\frac{f_0^2}{m\omega^3}\sin^2\frac{\omega T}{2} \tag{26}$$

Since this is a real quantity, we have, from 6

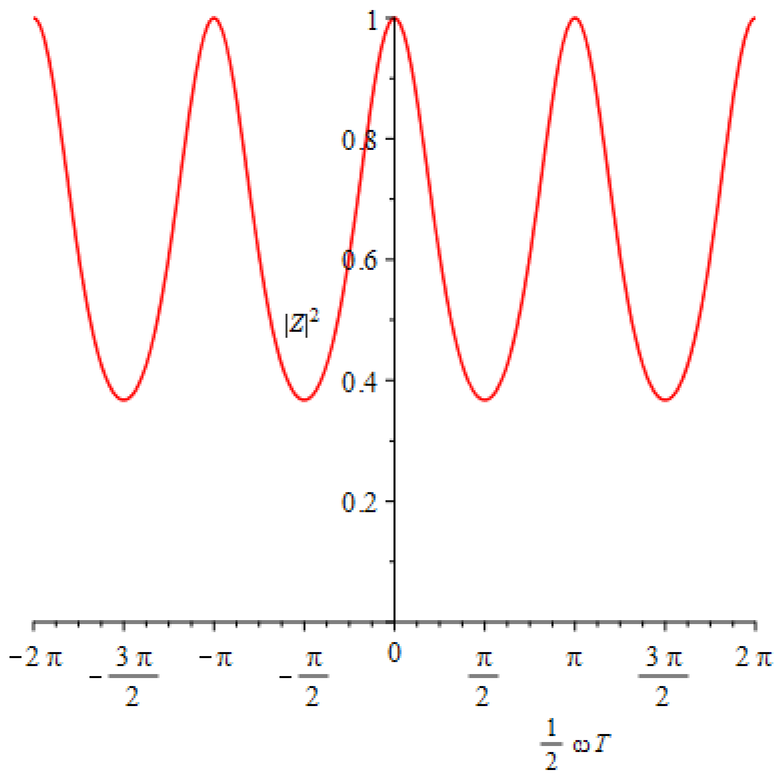

FIGURE 2. $|Z[f(t)]|^2$

$$Z[f(t)] = \exp\left(-\frac{f_0^2}{m\omega^3}\sin^2\frac{\omega T}{2}\right) \tag{27}$$

so that

$$|Z[f(t)]|^2 = \exp\left(-\frac{2f_0^2}{m\omega^3}\sin^2\frac{\omega T}{2}\right) \tag{28}$$

Fig. 2 shows a plot of $|Z[f(t)]|^2$ versus $\frac{\omega T}{2}$ for $\frac{2f_0^2}{m\omega^3} = 1$. The probability of zero particles being output has a maximum of 1 at $T = 0$ (when the forcing function is completely absent) and then oscillates between a minimum of around 0.4 and a maximum of 1. The minima occur at $T = (2n+1)\pi/\omega$.

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