**SCALAR FIELD THEORY WITH SOURCE FIELD**

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Reference: Tom Lancaster and Stephen J. Blundell, *Quantum Field Theory for the Gifted Amateur*, (Oxford University Press, 2014), Problem 22.2.

Post date: 10 Aug 2019.

Consider a scalar field with a source field $J(x)$ with a Lagrangian density

$$\mathcal{L}=\frac{1}{2}\left[\partial_{\mu}\phi(x)\right]^{2}-\frac{m^{2}}{2}\phi^{2}(x)+gJ(x)\phi(x) \tag{1}$$

One diagram contributing to the $S$ matrix is the dumbbell in Fig. 1.

In this diagram, the source acts at spacetime points $x$ and $y$, with a scalar propagator connecting these points. The rules for constructing the amplitude for this diagram in position space are similar to those for the $\phi^4$ theory given in L&B, chapter 19 (page 182). Each vertex contributes a factor of $-igJ$ and the propagator is the same as that for the scalar field theory given in L&B equation 17.24. Since the two ends of the dumbbell can be swapped, we have a symmetry factor of 2. Putting it all together we have

$$(\text{Dumbbell})=\frac{(-ig)^{2}}{2}\int d^{4}x\,d^{4}y\int\frac{d^{4}p}{(2\pi)^{4}}J(x)\frac{ie^{-ip\cdot(x-y)}}{p^{2}-m^{2}+i\epsilon}J(y) \tag{2}$$

We now consider a source field consisting of two static point sources at spatial locations $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, given by

$$J(x)=\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}_1)+\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}_2) \tag{3}$$

Inserting this into 2 we have

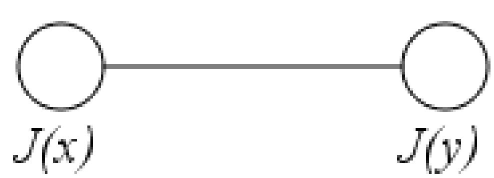


FIGURE 1. Dumbbell diagram.

$$(\text{Dumbbell}) = -\frac{ig^2}{2}\int d^4x\, d^4y \int \frac{d^4p}{(2\pi)^4}\left[\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}_1)+\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}_2)\right]\times \frac{ie^{-ip\cdot(x-y)}}{p^2-m^2+i\epsilon}\left[\delta^{(3)}(\boldsymbol{y}-\boldsymbol{x}_1)+\delta^{(3)}(\boldsymbol{y}-\boldsymbol{x}_2)\right] \tag{4}$$

$$= -\frac{ig^2}{2}\int d^4x\, d^4y \int \frac{d^4p}{(2\pi)^4}\left[\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}_1)\delta^{(3)}(\boldsymbol{y}-\boldsymbol{x}_1)+\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}_2)\delta^{(3)}(\boldsymbol{y}-\boldsymbol{x}_2)+\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}_1)\delta^{(3)}(\boldsymbol{y}-\boldsymbol{x}_2)+\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}_2)\delta^{(3)}(\boldsymbol{y}-\boldsymbol{x}_1)\right]\frac{e^{-ip\cdot(x-y)}}{p^2-m^2+i\epsilon} \tag{5}$$

If we ignore self-interaction terms, which are represented by the terms $\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}_1)\delta^{(3)}(\boldsymbol{y}-\boldsymbol{x}_1)$ and $\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}_2)\delta^{(3)}(\boldsymbol{y}-\boldsymbol{x}_2)$ where both delta functions act on the same spatial point, we are left with

$$(\text{Dumbbell}) = -\frac{ig^2}{2}\int d^4x\, d^4y \int \frac{d^4p}{(2\pi)^4}\left[\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}_1)\delta^{(3)}(\boldsymbol{y}-\boldsymbol{x}_2)+\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}_2)\delta^{(3)}(\boldsymbol{y}-\boldsymbol{x}_1)\right]\frac{e^{-ip\cdot(x-y)}}{p^2-m^2+i\epsilon} \tag{6}$$

We can now separate out the integral over the time components $x^0$ and $y^0$ and apply the delta functions to the spatial components. After doing the integrals over $d^3x$ and $d^3y$ we have:

$$(\text{Dumbbell}) = -\frac{ig^2}{2}\int dx^0\, dy^0 \int \frac{dp^0}{2\pi}\left[e^{-ip^0\left(x^0-y^0\right)}\times \int \frac{d^3p}{(2\pi)^3}\left(\frac{e^{-i\boldsymbol{p}\cdot(\boldsymbol{x}_2-\boldsymbol{x}_1)}}{p^2-m^2+i\epsilon}+\frac{e^{i\boldsymbol{p}\cdot(\boldsymbol{x}_2-\boldsymbol{x}_1)}}{p^2-m^2+i\epsilon}\right)\right] \tag{7}$$

The two terms in the $d^3p$ integral can be combined by making the substitution $\boldsymbol{p}\to-\boldsymbol{p}$ in the second term. This doesn't affect the $p^2=\left(p^0\right)^2-\boldsymbol{p}^2$ term in the denominator. The integration limits are reversed which gives rise to a net minus sign, but this cancelled by the integration measure going from $d^3p\to-d^3p$. Thus we are left with

$$
\begin{aligned}
(\text{Dumbbell}) = -\frac{ig^2}{2}\int dx^0\, dy^0 \int \frac{dp^0}{2\pi}\Big[e^{-ip^0\left(x^0-y^0\right)} \times \\
\int \frac{d^3p}{(2\pi)^3}\left(\frac{e^{i\boldsymbol{p}\cdot(\boldsymbol{x}_1-\boldsymbol{x}_2)}}{p^2-m^2+i\epsilon}+\frac{e^{i\boldsymbol{p}\cdot(\boldsymbol{x}_1-\boldsymbol{x}_2)}}{p^2-m^2+i\epsilon}\right)\Big] \quad (8)\\
= -ig^2\int dx^0\, dy^0\int\frac{dp^0}{2\pi}e^{-ip^0\left(x^0-y^0\right)}\int\frac{d^3p}{(2\pi)^3}\frac{e^{i\boldsymbol{p}\cdot(\boldsymbol{x}_1-\boldsymbol{x}_2)}}{p^2-m^2+i\epsilon} \quad (9)
\end{aligned}
$$

Notice that the integration over $p^0$ includes the last integral over $d^3p$ since the denominator contains a $\left(p^0\right)^2$ term.

We can now do the integral over $y^0$ since the only place $y^0$ occurs is in the first exponent. We have

$$\int\frac{dy^0}{2\pi}e^{-ip^0y^0} = \delta\left(p^0\right) \quad (10)$$

so we get

$$
\begin{aligned}
(\text{Dumbbell}) &= -ig^2\int dx^0 dp^0 e^{-ip^0x^0}\int\frac{d^3p}{(2\pi)^3}\frac{e^{i\boldsymbol{p}\cdot(\boldsymbol{x}_1-\boldsymbol{x}_2)}}{\left(p^0\right)^2-\boldsymbol{p}^2-m^2+i\epsilon} && (11)\\
&= -ig^2\int dx^0\int\frac{d^3p}{(2\pi)^3}\frac{e^{i\boldsymbol{p}\cdot(\boldsymbol{x}_1-\boldsymbol{x}_2)}}{-\boldsymbol{p}^2-m^2+i\epsilon} && (12)\\
&= ig^2\int dx^0\int\frac{d^3p}{(2\pi)^3}\frac{e^{i\boldsymbol{p}\cdot(\boldsymbol{x}_1-\boldsymbol{x}_2)}}{\boldsymbol{p}^2+m^2-i\epsilon} && (13)
\end{aligned}
$$

Taking the limit $\epsilon\to 0$ gives us

$$(\text{Dumbbell}) = ig^2\int dx^0\int\frac{d^3p}{(2\pi)^3}\frac{e^{i\boldsymbol{p}\cdot(\boldsymbol{x}_1-\boldsymbol{x}_2)}}{\boldsymbol{p}^2+m^2} \quad (14)$$

As with the forced harmonic oscillator we have

$$e^{-iET} = e^{(\text{Dumbbell})} \quad (15)$$

where $T$ is the time over which the sources act. In 14, the time appears only as the integration variable $x^0$, so we have

$$(\text{Dumbbell}) = ig^2T\int\frac{d^3p}{(2\pi)^3}\frac{e^{i\boldsymbol{p}\cdot(\boldsymbol{x}_1-\boldsymbol{x}_2)}}{\boldsymbol{p}^2+m^2} \quad (16)$$

Comparing this with 15 we have

$$-iET = ig^2T\int\frac{d^3p}{(2\pi)^3}\frac{e^{i\boldsymbol{p}\cdot(\boldsymbol{x}_1-\boldsymbol{x}_2)}}{\boldsymbol{p}^2+m^2} \tag{17}$$

$$E = -g^2\int\frac{d^3p}{(2\pi)^3}\frac{e^{i\boldsymbol{p}\cdot(\boldsymbol{x}_1-\boldsymbol{x}_2)}}{\boldsymbol{p}^2+m^2} \tag{18}$$

[I'll leave the final part (e) of this question until I've covered the relevant material.]