WICK'S THEOREM AND PATH INTEGRALS

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Reference: Tom Lancaster and Stephen J. Blundell, *Quantum Field Theory for the Gifted Amateur*, (Oxford University Press, 2014), Problem 23.2.

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We start with the Gaussian integral

$$I_{2n}(a)=\int_{-\infty}^{\infty}x^{2n}e^{-\frac{1}{2}ax^2}dx \tag{1}$$

We can evaluate this by taking the derivative with respect to a, as in

$$\begin{aligned} I_2(a) &= -2\frac{\partial}{\partial a}I_0(a) && (2)\\ &= \frac{\sqrt{2\pi}}{a^{3/2}} && (3) \end{aligned}$$

where we've used the standard result for a Gaussian integral

$$\int_{-\infty}^{\infty}e^{-\frac{1}{2}ax^2}dx=\sqrt{\frac{2}{a}}\int_{-\infty}^{\infty}e^{-u^2}du=\sqrt{\frac{2\pi}{a}} \tag{4}$$

The Gaussian mean of a power of x is defined by

$$\langle x^n\rangle=\frac{\int_{-\infty}^{\infty}x^n e^{-\frac{1}{2}ax^2}dx}{\int_{-\infty}^{\infty}e^{-\frac{1}{2}ax^2}dx}$$

We therefore have

$$\left\langle x^2\right\rangle=\frac{1}{a} \tag{5}$$

We can continue to find higher powers of $\langle x^n\rangle$ by successive derivatives, but I just used Maple to get the results:

All odd powers of $\left\langle x^{2n+1}\right\rangle$ are zero, since x^{2n+1} is an odd function integrated over a symmetric interval.

$$\begin{aligned} \left\langle x^4\right\rangle &= \frac{3}{a^2} && (6)\\ \left\langle x^6\right\rangle &= \frac{15}{a^3} && (7)\\ \left\langle x^8\right\rangle &= \frac{105}{a^4} && (8) \end{aligned}$$

We can spot a pattern here and guess that

$$\left\langle x^{2n}\right\rangle = \frac{(2n-1)(2n-3)\ldots3\times1}{a^{n}} = \frac{(2n-1)!!}{a^{n}} \tag{9}$$

We can verify this using Maple, which gives

$$\left\langle x^{2n}\right\rangle = \frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}}\frac{2^{n}}{a^{n}} \tag{10}$$

The gamma function has the value (see, for example, the Wikipedia article):

$$\Gamma\left(n+\frac{1}{2}\right) = \frac{(2n)!}{4^{n}n!}\sqrt{\pi} \tag{11}$$

$$= \frac{(2n)(2n-1)(2n-2)\ldots(2)(1)}{4^{n}n!}\sqrt{\pi} \tag{12}$$

$$= \frac{2^{n}}{4^{n}}\frac{n!(2n-1)!!}{n!}\sqrt{\pi} \tag{13}$$

$$= \frac{(2n-1)!!\sqrt{\pi}}{2^{n}} \tag{14}$$

Plugging this back into 10 we have the desired result:

$$\left\langle x^{2n}\right\rangle = \frac{(2n-1)!!}{a^{n}} \tag{15}$$

or, for n even:

$$\left\langle x^{n}\right\rangle = \frac{(n-1)!!}{a^{n/2}} \tag{16}$$

In the next part of the problem, we're asked to represent this result diagrammatically by using each factor of $\frac{1}{a}$ as a link between a pair of factors of x. I think the idea here is to draw an analogy with Wick's theorem. We always have half the number of $\frac{1}{a}$ factors as we have factors of x, so presumably the idea is that we link each pair of separate factors of x by a line represented by $\frac{1}{a}$. For a single pair of factors of x, we can do this in only one way, so as we see in 5, $\left\langle x^{2}\right\rangle = \frac{1}{a}$. For four factors of x, we can link the first x with each of the remaining 3. In each case, the other two factors can be linked in only one way. Thus we have 2 lines connecting the 2 pairs, and this can be done in 3 ways, giving $\left\langle x^{4}\right\rangle = \frac{3}{a^{2}}$. For higher powers of x the same argument holds. For $\left\langle x^{n}\right\rangle$ we can link x_{1} with $n-1$ other nodes, which leaves $n-2$ other nodes to link up, and these $n-2$ nodes can be linked up according to the factor $\left\langle x^{n-2}\right\rangle$, which we had already worked out. Thus, by induction, we have

$$\langle x^n \rangle = \frac{(n-1)}{a} \left\langle x^{n-2} \right\rangle \tag{17}$$

We now consider the integral

$$\mathcal{K} = \int dx_1 \ldots dx_N \; e^{-\frac{1}{2}\boldsymbol{x}^T \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}^T \boldsymbol{x}} \tag{18}$$

where $\boldsymbol{A}$ is a symmetric matrix and $\boldsymbol{b}$ is a constant vector. In Chapter 23, L&B show that this integral is given by equation 23.27:

$$\mathcal{K} = \left(\frac{(2\pi)^N}{\det \boldsymbol{A}} \right)^{1/2} e^{\frac{1}{2}\boldsymbol{b}^T \boldsymbol{A}^{-1} \boldsymbol{b}} \tag{19}$$

We consider

$$\langle x_i x_j \rangle = \frac{\int dx_1 \ldots dx_N \; x_i x_j e^{-\frac{1}{2}\boldsymbol{x}^T \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}^T \boldsymbol{x}}}{\int dx_1 \ldots dx_N \; e^{-\frac{1}{2}\boldsymbol{x}^T \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}^T \boldsymbol{x}}} \tag{20}$$

The integral in the numerator can be found by differentiating 18 with respect to b_i and then b_j and then setting $\boldsymbol{b} = \boldsymbol{0}$. We can do this by using the result 19. We have

$$\frac{\partial}{\partial b_i} \mathcal{K} = \left(\frac{(2\pi)^N}{\det \boldsymbol{A}} \right)^{1/2} e^{\frac{1}{2}\boldsymbol{b}^T \boldsymbol{A}^{-1} \boldsymbol{b}} \left[\frac{1}{2} A_{ik}^{-1} b_k + \frac{1}{2} b_k A_{ki}^{-1} \right] \tag{21}$$

$$\frac{\partial}{\partial b_j \partial b_i} \mathcal{K} = \left(\frac{(2\pi)^N}{\det \boldsymbol{A}} \right)^{1/2} e^{\frac{1}{2}\boldsymbol{b}^T \boldsymbol{A}^{-1} \boldsymbol{b}} \left[\frac{1}{2} A_{jk}^{-1} b_k + \frac{1}{2} b_k A_{kj}^{-1} + \frac{1}{2} A_{ij}^{-1} + \frac{1}{2} A_{ji}^{-1} \right] \tag{22}$$

Setting $\boldsymbol{b} = \boldsymbol{0}$ we have

$$\left. \frac{\partial}{\partial b_j \partial b_i} \mathcal{K} \right|_{\boldsymbol{b}=\boldsymbol{0}} = \left(\frac{(2\pi)^N}{\det \boldsymbol{A}} \right)^{1/2} \left[\frac{1}{2} A_{ij}^{-1} + \frac{1}{2} A_{ji}^{-1} \right] \tag{23}$$

$$= \left(\frac{(2\pi)^N}{\det \boldsymbol{A}} \right)^{1/2} A_{ij}^{-1} \tag{24}$$

where the last step follows because $\boldsymbol{A}$ and thus $\boldsymbol{A}^{-1}$ is symmetric. Plugging this and 19 (also with $\boldsymbol{b} = \boldsymbol{0}$) into 20 we have

$$\langle x_i x_j \rangle = A_{ij}^{-1} \tag{25}$$

I think the idea behind the remainder of the problem is that we take $\langle x_i x_j \rangle$ to be the value of a Wick 'contraction' between nodes x_i and x_j in analogy to the situation above with a single variable x. Using this idea, the expression for $\langle x_i x_j x_k x_\ell \rangle$ is constructed by considering all possible contractions among the four nodes. We can have x_i contracted with x_j (which then requires that x_k contracts with x_ℓ), or x_i contracted with x_k (which then requires that x_j contracts with x_ℓ), or finally, x_i contracted with x_ℓ (which then requires that x_j contracts with x_k). This gives the result

$$\langle x_i x_j x_k x_\ell \rangle = A_{ij}^{-1} A_{k\ell}^{-1} + A_{ik}^{-1} A_{j\ell}^{-1} + A_{i\ell}^{-1} A_{kj}^{-1} \tag{26}$$

Generalizing this idea to an arbitrary (but even) number of nodes $x_i \ldots x_z$, we see that the nodes must contract in pairs, and we must consider all possible pairings. There will be a term $A_{ij}^{-1} A_{k\ell}^{-1} \ldots A_{yz}^{-1}$ for each possible set of pairings. If there are $2n$ nodes (values of x_i), there will be $(2n-1)!!$ terms in the sum, with each term consisting of n factors of A_{ij}^{-1}. For example, for $2n = 6$, each term contains 3 factors of A_{ij}^{-1} and there are $(6-1)!! = 15$ terms in the sum.