TRANSITION AMPLITUDE FOR FORCED HARMONIC OSCILLATOR

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Reference: Tom Lancaster and Stephen J. Blundell, *Quantum Field Theory for the Gifted Amateur*, (Oxford University Press, 2014), Problem 23.3.

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We revisit the forced harmonic oscillator with Lagrangian

$$L=\frac{1}{2}m\dot{x}^{2}(t)-\frac{1}{2}m\omega^{2}x^{2}(t)+f(t)\,x(t) \tag{1}$$

Here $f(t)$ is the forcing function which has the constant value f_0 for $0\le t\le T$ and zero elsewhere.

As we saw in the earlier post, the amplitude for a particle starting in the ground state at $t=0$ to also be in the ground state at $t=T$ is

$$\mathcal{A}=\exp\left[-\frac{1}{2}\int dt\,dt'f(t)\left\langle 0\left|\mathscr{T}x(t)\,x\left(t'\right)\right|0\right\rangle f\left(t'\right)\right] \tag{2}$$

The term $\langle 0|\mathscr{T}x(t)\,x(t')|0\rangle$ is the propagator from time t' to t (or vice versa, depending on the time ordering operator $\mathscr{T}$). In momentum space, this propagator is

$$\tilde{G}(\nu)=\frac{i}{m\left(\nu^{2}-\omega^{2}+i\epsilon\right)} \tag{3}$$

In order to write $\mathcal{A}$ in 2, we need G in the time domain instead of the frequency domain. There might be an easier way to get this, but the following is the best I could find.

We need the Fourier transform of 3, which is

$$\left\langle 0\left|\mathscr{T}x(t)\,x\left(t'\right)\right|0\right\rangle=G\left(t,t'\right)=\frac{1}{2\pi}\int d\nu\;e^{-i\nu(t-t')}\tilde{G}(\nu) \tag{4}$$

$$=\frac{i}{2\pi m}\int d\nu\frac{e^{-i\nu(t-t')}}{\nu^{2}-\omega^{2}+i\epsilon} \tag{5}$$

We can do this integral by using partial fractions and then contour integration. We have, to first order in ϵ

$$\frac{1}{\nu^{2}-\omega^{2}+i\epsilon}=\frac{1}{2\omega}\left[\frac{1}{\nu-\omega+i\epsilon}-\frac{1}{\nu+\omega-i\epsilon}\right] \tag{6}$$

We have split 5 into two separate integrals. The first one is

$$G_1 = \frac{i}{4\pi m\omega}\int d\omega \frac{e^{-i\nu(t-t')}}{\nu-\omega+i\epsilon} \tag{7}$$

The integrand has a simple pole at $\nu = \omega - i\epsilon$, which is in the lower half plane. If we use the usual semi-circular contour with the edge going along the real axis and the semi-circular arc closing it in a clockwise direction in the lower half plane, then we get the required integral if we can show that the integral around the arc goes to zero as $|\nu| \to \infty$ on this arc. On the arc, we can represent ν by

$$\nu = r\cos\theta + ir\sin\theta \tag{8}$$

where $\pi \le \theta \le 2\pi$. In this region, $\sin\theta \le 0$. The exponential in the integrand in 7 is thus

$$e^{-i\nu(t-t')} = \exp\left[-ir\left(t-t'\right)\cos\theta + r\left(t-t'\right)\sin\theta\right] \tag{9}$$

Since $\sin\theta \le 0$, we must have $t - t' > 0$ in order for the real part of the exponent to be negative, which is what's needed in order for the overall exponential to go to zero as $r \to \infty$.

The residue at the pole is

$$\text{Res} = \lim_{\nu\to\omega-i\epsilon}\left(\nu-\omega+i\epsilon\right)\frac{e^{-i\nu(t-t')}}{\nu-\omega+i\epsilon} \tag{10}$$

$$= e^{-i\omega(t-t')} \tag{11}$$

where we've also taken the limit $\epsilon \to 0$ since it is no longer required to make the integral converge.

The integral in 7 is therefore $2\pi i$ times the residue, so we have

$$G_1 = -\frac{2\pi i^2}{4\pi m\omega}e^{-i\omega(t-t')}\theta\left(t-t'\right) \tag{12}$$

$$= \frac{1}{2m\omega}e^{-i\omega(t-t')}\theta\left(t-t'\right) \tag{13}$$

The minus sign at the front comes from taking a clockwise contour in the lower half plane.

where $\theta\left(t-t'\right)$ is the usual step function which imposes the condition $t > t'$.

The other integral arising from the second partial fraction in 6 is

$$G_2 = -\frac{i}{4\pi m\omega}\int d\omega \frac{e^{-i\nu(t-t')}}{\nu+\omega-i\epsilon} \tag{14}$$

This time, the pole is in the upper half plane at $\nu = -\omega + i\epsilon$, so we use a counterclockwise contour. In this half plane, $\sin\theta$ in 8 is positive, so in order for the real part of the exponent in 9 to be negative, we now require $t < t'$. The residue is

$$\text{Res} = \lim_{\nu \to -\omega + i\epsilon} (\nu + \omega - i\epsilon) \frac{e^{-i\nu(t-t')}}{\nu + \omega - i\epsilon} \tag{15}$$

$$= e^{i\omega(t-t')} \tag{16}$$

and the overall integral comes out to

$$G_2 = \frac{1}{2m\omega} e^{i\omega(t-t')} \theta\left(t' - t\right) \tag{17}$$

The complete integral is then

$$G\left(t, t'\right) = G_1 + G_2 \tag{18}$$

$$= \frac{1}{2m\omega} \left(e^{-i\omega(t-t')} \theta\left(t - t'\right) + e^{i\omega(t-t')} \theta\left(t' - t\right) \right) \tag{19}$$

We can now do the integral to get $\mathcal{A}$ from 2. For G_1 we have

$$\int G_1 = \frac{1}{2m\omega} \int_0^T dt \int_0^t dt'\, e^{-i\omega(t-t')} \tag{20}$$

$$= \frac{1}{i\omega} \frac{1}{2m\omega} \int_0^T dt \left(1 - e^{-i\omega t}\right) \tag{21}$$

$$= -\frac{i}{2m\omega^2} \left(T + \frac{1}{i\omega} \left(e^{-i\omega T} - 1 \right) \right) \tag{22}$$

$$= -\frac{1}{2m\omega^2} \left(iT + \frac{1}{\omega} \left(e^{-i\omega T} - 1 \right) \right) \tag{23}$$

For G_2 we have

$$\int G_2 = \frac{1}{2m\omega} \int_0^T dt \int_t^T dt' e^{i\omega(t-t')} \tag{24}$$

$$= -\frac{1}{i\omega} \frac{1}{2m\omega} \int_0^T dt \left(e^{i\omega(t-T)} - 1 \right) \tag{25}$$

$$= \frac{i}{2m\omega^2} \left[\frac{1}{i\omega} \left(1 - e^{-i\omega T} \right) - T \right] \tag{26}$$

$$= -\frac{1}{2m\omega^2} \left[iT + \frac{1}{\omega} \left(e^{-i\omega T} - 1 \right) \right] \tag{27}$$

The complete integral is therefore

$$G(0,T) = \int G_1 + \int G_2 \tag{28}$$

$$= -\frac{1}{m\omega^2}\left[iT + \frac{1}{\omega}\left(e^{-i\omega T} - 1\right)\right] \tag{29}$$

We can convert this to trig functions as follows.

$$e^{-i\omega T} - 1 = e^{-i\omega T/2}\left(e^{-i\omega T/2} - e^{i\omega T/2}\right) \tag{30}$$

$$= -2i\left(\cos\frac{\omega T}{2} - i\sin\frac{\omega T}{2}\right)\sin\frac{\omega T}{2} \tag{31}$$

$$= -2i\sin\frac{\omega T}{2}\cos\frac{\omega T}{2} - 2\sin^2\frac{\omega T}{2} \tag{32}$$

$$= -i\sin\omega T - 2\sin^2\frac{\omega T}{2} \tag{33}$$

Therefore

$$G(0,T) = -\frac{i}{m\omega^2}\left[T - \frac{\sin\omega T}{\omega} + \frac{2i}{\omega}\sin^2\frac{\omega T}{2}\right] \tag{34}$$

The two factors of $f(t)$ and $f(t')$ in 2 each contribute a constant factor of f_0 since the forcing function is constant over $0 \le t \le T$, so inserting 34 into 2 gives us

$$\mathcal{A} = \exp\left[\frac{if_0^2}{2m\omega^2}\left(T - \frac{\sin\omega T}{\omega} + \frac{2i}{\omega}\sin^2\frac{\omega T}{2}\right)\right] \tag{35}$$

The probability for the oscillator to start and end in the ground state is therefore

$$|\mathcal{A}|^2 = \exp\left(-\frac{2f_0^2}{m\omega^3}\sin^2\frac{\omega T}{2}\right) \tag{36}$$

which agrees with the earlier calculation.

I'm not sure what the physical meaning of the imaginary part of the exponent in 35 is. Usually an imaginary component of an amplitude is interpreted as a phase factor which cancels out when we calculate the probability, as it does here.